Unsaturated Zone Hydrology for Scientists and Engineers

James A. Tindall, Ph.D.

United States Geological Survey, National Research Program
Department of Geography and Environmental Sciences, University of Colorado Denver

James R. Kunkel, Ph.D., P.E.

Knight Piésold, LLC, Denver, Colorado;
Department of Geology and Geological Engineering, Colorado School of Mines

with

Dean E. Anderson, Ph.D.

United States Geological Survey, National Research Program

Site Characterization and Monitoring Devices

INTRODUCTION

This appendix will acquaint the reader with the site-selection process, including the primary criteria used for selecting a site for waste storage or scientific investigation. We will also discuss the most common monitoring devices used in unsaturated-zone studies, their principles of operation, and the advantages and disadvantages of each.

SITE CHARACTERIZATION

The purpose of site characterization is to determine the biological, chemical, and physical properties at a site that directly affect the movement of contaminants from or within it. However, before site characterization can take place, a site must be selected. The site-selection process can involve a large number of criteria: the availability of land; climatological factors that may bias the outcome of potential accident scenarios; proximity to transportation and population centers; proximity to sensitive natural resources such as aquifers, prevailing winds; and so on. Ideally, the site should be capable of being characterized, analyzed, monitored, and modeled. Because of this, it would be prudent to select several candidate sites based on technical criteria, depending upon the intended use of the site. For example, the criteria necessary for installing a safe landfill would not be as stringent as that for installing a low-level radioactive waste site. In many instances, local, state, or federal agencies already will have established required site guidelines.

The site-selection process usually begins by establishing technical criteria with respect to depth to ground water, acceptable rainfall limits, slope, elevation, and so on. After these have been set, a large-scale reconnaissance of various geographic areas is performed. From these areas, candidate sites are chosen and screened against minimum technical requirements. Site selection can be facilitated by obtaining soil maps and information on the physical properties of soil series from the National Resources Conservation Service (previously the U.S. Soil Conservation Service), and ground water table information from the United States Geological Survey. Once the candidate sites have been screened and a final site selected, it is ready for characterization.

The goal of characterization is three-fold: (1) to identify potential pathways for the transport of contaminants from containment areas to sensitive receptors—drinking-water supplies, air, and so on; (2) to demonstrate that the site can be characterized, monitored, and modeled, which involves field and laboratory analysis; and (3) to confirm that

performance objectives (which comprise data analysis and modeling) can be met. Field characterization can be accomplished by in-situ testing, or by collecting disturbed and undisturbed samples for laboratory analysis. In-situ testing is usually preferred, but may not be practical for a large number of parameters. It is also time-intensive, especially for unsaturated-zone properties that occur at greatly reduced rates in comparison to the saturated-zone environment. When designing the characterization effort, the purpose for which the data to be collected will be used must be clear. For example, is the purpose simple data analysis and monitoring, or modeling the site in preparation for a license application to store hazardous waste of some type? Whatever the need, characterization efforts should be designed in cooperation with those who will use the data: regulatory agencies, modelers, and other personnel.

Field characterization can be grouped into the following broad categories: (1) chemical properties; (2) physical properties; (3) flow and transport properties; and (4) biological properties. Chemical properties of common interest include pH, chloride and sulfate concentrations, cation exchange capacity, and total salt concentration of the soil solution. Typical physical properties include bulk density, particle density, porosity, and particle-size analysis. Flow properties of interest are saturated hydraulic conductivity, unsaturated hydraulic conductivity (as a function of water content), and soil-moisture characteristic curves that relate water content to matric potential; the transport property of most interest is the dispersion coefficient, D (see chapter 10). Biological properties can include a host of microbiological properties and the properties discussed for physical and chemical parameters. Details on measurement techniques and analysis for each of these properties can be found in Klute (1986).

For modeling purposes, initial and boundary conditions must also be known. These include the vertical profiles over a depth of interest of water content and potential; the amount of water deposited on-site in the form of precipitation (rain and/or snow); and the amount of water leaving the site by overland flow, drainage, and evapotranspiration. For complete characterization, a representative number of samples for each parameter must be collected from the soil surface to the depth of interest (usually the water table). No definitive method has been presented that will provide absolute numbers or locations of samples for analysis, but there are equations that can be used in approximating the necessary number of samples for a chosen parameter (for example, equation 16.21 in this text). As with most scientific research and data collection needs, economics will play a major role in determining the number of samples that can be analyzed.

Frequently, environmental regulations require monitoring of the unsaturated zone for many sites. Ideally, one would wish to minimize cost yet assure reliability, which requires that monitoring devices be installed in the best possible locations. Additionally, the sample volume for most site-selection (or other investigative) purposes is small in comparison to the spacing interval from which the sample is taken. Because of this, there is a significant probability that any anomalies or "hot spots" will not be detected, and such areas are where one would most wish to sample. These are areas where physical, chemical, biological, and flow characteristics can be very different from those that exist in the site as a whole; and spots where a contaminant may be more prone to leak. In order to detect such areas, specific questions must be asked about the site at the initial planning and installation stages: (1) what grid spacing is needed to hit a hot spot with a specified confidence; (2) what is the probability that a hot spot exists when none are found by sampling on a grid; and (3) for a given grid spacing, what is the probability of hitting a hot spot of a specified size? Other questions may also arise, but these are the most important. For further details, the reader is referred to Warrick et al. (1996).

MONITORING DEVICES

This section discusses factors that influence the choice of monitoring devices, and the common types of equipment used for monitoring various soil parameters in unsaturated-zone studies, as well as various measurement techniques. The basic suitability, advantages, and disadvantages of each piece of equipment will be briefly outlined.

Factors Influencing Choice of Devices

The basic factors that influence the choice of monitoring devices include: **(1)** goals of the monitoring program; **(2)** site environment and conditions; **(3)** repeatability of measurements; **(4)** measurement-device resolution and operational range; **(5)** equipment and device durability; **(6)** device installation and replacement; **(7)** remote data acquisition versus manual data collection; and **(8)** device maintenance frequency.

Goals of the monitoring program may include process, compliance or remediation monitoring. *Process monitoring* is commonly used to determine physical processes for establishing background and baseline conditions. This may require high-intensity monitoring near the soil surface to understand periodic changes in data because of seasonal fluctuations. During process monitoring, goals are commonly achieved by strong reliance on data acquisition systems capable of sustained, high-frequency sampling. *Compliance monitoring* is generally performed to confirm waste isolation during operational and post-closure monitoring programs. Specific goals are normally to test the behavior of various parameters and compare them to baseline studies. This will ascertain whether contaminant migration or other significant events are occurring. Frequent data collection may be decreased to daily, weekly, or quarterly intervals depending on the individual parameter. As intervals become less frequent, automated systems may be replaced by manual data collections without accuracy loss. *Remediation monitoring* tracks the success of any cleanup activities that have occurred, and can be short- or long-term depending on the site. This type of monitoring usually combines procedures listed in process and compliance monitoring.

Site and environmental conditions include the soil type and structure; the presence of rocks; layering; depth to water table; macropores; and any obstruction that would affect instrument installation. Seasonally flooded or arid conditions will influence the choice of instrumentation, as well as the depth to the water table. Proximity to a city or town, as well as the availability of AC power should be considered part of the site conditions. This affects the economics, and the choice of manual or automated data sampling and acquisition. The presence of AC power increases the flexibility of instruments that can be used on-site.

Any monitoring study, regardless of duration, must have the ability to obtain repeatable data measurements. These might include electrical conductivity, soil dielectric constant, soil-water energy status, pH, and other parameters. The devices used must be able to obtain reliable, repeatable results.

Measurement-device resolution and operational range must be sufficient to detect changes in soil conditions. The accuracy and precision of each measuring device must be determined, and matched to monitoring goals. The operational range of each device can be used to determine its suitability in the overall monitoring strategy; for example, if the soil is very dry, tensiometers would not be as suitable for measuring soil-water energy status as thermocouple psychrometers. Devices with very narrow measurement ranges should not be relied upon in a long-term monitoring program, unless site conditions are well-understood.

Device life-span, and use in long-term systems, must be considered; that is, whether or not it will extend into post-operational or long-term care periods. If extension into these periods is not possible, the device will only be suitable for establishing baseline conditions.

Devices must be compatible with the specific phase of monitoring, operational or post-operational. Devices or locations where failure could affect the overall integrity of the monitoring program should be avoided.

Because the method of device installation and replacement can affect the measurements of water movement and other parameters, it is important to assess the advantages and disadvantages of horizontal versus vertical installation for each device. The ability to remove, repair, and replace equipment as necessary is an important factor in determining initial installation geometry. For example, at a site where accessibility is easy and long-term, a network of nested devices can be employed by installing them in a caisson of specific diameter and length (see figure A1.1). Depending on site accessibility, installing extra data-access ports adjacent to the site can provide flexibility in the event of instrument failure, the need for replacement, or difficult access after the site is closed.

Remote data acquisition can save valuable time and resources compared to manual data collection. The economics of the labor costs of manual collection versus the higher maintenance costs of telephone modems, solar panels, and computer manipulation must be assessed. Quite often, the maintenance and recalibration of data acquisition systems is complicated by the need to remove a storage module or data-logging device from the field, and return it to the laboratory. However, for various devices capable of automation, remote data acquisition is a very favorable option—although manual data collection allows technical personnel to check on-site equipment and make necessary adjustments. The decision to remotely access a site will depend on the type of data being collected, the frequency that it is needed, the site accessibility, the ability of the remotely accessible device to be incorporated into the overall monitoring scheme, and other factors.

Device maintenance frequency is an important aspect of site monitoring, as high maintenance requirements can affect the long-term viability of the monitoring system and goals, as well as accuracy and personnel costs. As a result, devices that are actively maintained
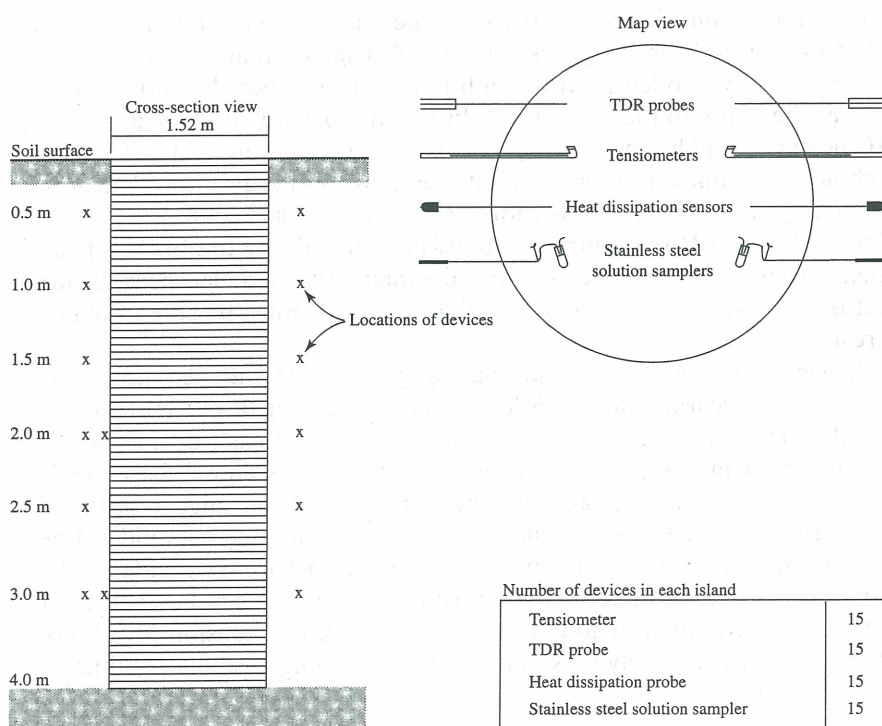

Figure A1.1   Diagram of monitoring island with proposed instrumentation. The caisson shown in the cross-section view is constructed of corrugated culvert pipe. Data from Young et al. (1996)

Number of devices in each island	
Tensiometer	15
TDR probe	15
Heat dissipation probe	15
Stainless steel solution sampler	15

(such as tensiometers), and devices that require passive maintenance (such as pressure transducers), are not as useful for long-term monitoring. The overall durability and maintenance requirements for each device must be considered for the monitoring system and goals to be successful.

Water Content Measuring Devices

Neutron probe   The neutron probe works on the principle of neutron thermalization. It determines water content by releasing high-energy ("fast") neutrons from a radioactive source, such as americium-beryllium. The high-energy neutrons collide with hydrogen atoms in the soil, and form what are known as thermal neutrons. The multiple collisions that take place form a thermal cloud of neutrons whose size is constant, but whose density is dependent on soil-water content. Higher water content leads to increased thermalization and, thus, denser thermal clouds. A "slow" neutron detector, installed adjacent to the source emitter, measures the cloud. The measurement is displayed in the form of a "count ratio," with a higher count denoting a higher water content.

Suitability Criteria: **(1)** possible to merge with other devices; **(2)** durable (greater than 30 years experience); **(3)** horizontal and vertical installation possible; **(4)** after initial tube installation further monitoring scheme disturbance is not necessary.

Advantages and Disadvantages: **(1)** widely accepted with highly reproducible results; **(2)** contains a source of radioactivity and cannot be automated; **(3)** expensive (approximately $8500); and **(4)** requires calibration. Best method is to vertically install neutron tube adjacent to site, take measurements at specific intervals, then destructively obtain three samples directly around the tube in pyramid fashion at each measurement interval. Once this has been accomplished, the samples can be taken back to the laboratory, placed in an oven, and dried at 105 °C for twenty-four hours. A regression equation can then be obtained from the gravimetric water content of the samples versus the count ratio from the neutron probe for each interval (instructors: see final exam in solution manual).

Time domain reflectometry (TDR)   TDR operates on the principle of microwave-pulse travel through a parallel transmission line (the probe). This technology was adapted from the electric power industry, where cable testers are used to determine the location of a break in a power line. The speed of the microwave pulse depends on the dielectric constant K of the medium that surrounds and is in contact with the probe. Because of the significant difference between the dielectric constant of water and those of other constituents in soils, the speed of the pulse down the probe is highly dependent on soil-water content. When the microwave pulse reaches the end of the probe, the remaining energy is reflected back through the line. The apparent dielectric constant Ka of the soil can then be determined by

$$Ka = \left(\frac{tc}{L}\right)^2$$

where L is the length of the probe or "wave guides" (cm), t is the transit time (nanoseconds), and c is the speed of light (cm/nanosecond). The transit time is defined as the time required for the pulse to travel the length of the probe. Depending on the TDR unit, Ka is either calculated manually or, in some units, internally by the use of a zero/reset button and a visual display on the unit panel.

Suitability Criteria: **(1)** possible to merge with other devices; **(2)** durable; **(3)** horizontal and vertical installation possible with some exceptions; and **(4)** after initial tube installation, further monitoring scheme disturbance is not necessary.

Advantages and Disadvantages: **(1)** rapid, reliable, and repeatable; **(2)** minimal soil disturbance; **(3)** probe installation at any orientation; **(4)** no calibration necessary for rough

estimates of water content; **(5)** may be automated; **(6)** no radioactive source needed; **(7)** calibration necessary for accurate water content values; **(8)** cable length limited to approximately 50 m; **(9)** expensive ($8500 for cable tester alone); **(10)** has not been in use for more than fifteen years; and **(11)** automation difficult to maintain in field conditions.

Electromagnetic induction (EMI)   An EMI device does not measure water content directly, but measures soil electrical conductivity ($mS\ m^{-1}$). Readings are taken manually, or a data logger is incorporated for multiple readings and storage. Once data is collected, it is commonly entered into a spreadsheet program or imported into a database. Water content is then estimated through changes in electrical conductivity. In actuality, the device indicates the position of the wetting front. The greatest benefit of EMI is to quickly show the wetting front to depths of 1–2 m, and to detect anomalies in the soil profile, such as incongruities due to rock or heavy metals, and so on. Therefore, EMI can be an invaluable tool in the final site-selection and characterization process.

An EMI device contains a transmitter coil that induces circular eddy current loops in the soil. The magnitude associated with a loop is directly proportional to the electrical conductivity of the soil around the area of the loop. Each loop generates a secondary electromagnetic field (emf) that is also proportional to the current flowing in the loop. The emf induced by each loop is intercepted by a receiver near the transmitter, which amplifies and forms the emf into an output voltage that is linearly related to soil electrical conductivity. An example of an EMI plot is shown in figure A1.2.

Suitability Criteria: **(1)** possible to merge with other devices; **(2)** durable (several decades experience); and **(3)** no installation required (i.e., nondestructive).

Advantages and Disadvantages: **(1)** rapid, reliable, and repeatable; **(2)** no disturbance; **(3)** only works from soil surface; **(4)** calibration necessary (device must be zeroed at least once for each measurement day); **(5)** cannot be automated; **(6)** no radioactive source needed; **(7)** does not give water content directly; **(8)** expensive (approximately $6500).

Electrical resistivity borehole tomography (ERBT)   ERBT is not a new concept, but is an extention of surface resistivity. However, technological advances in electronics have provided a valuable tool for long-term monitoring. An electrical current is passed from
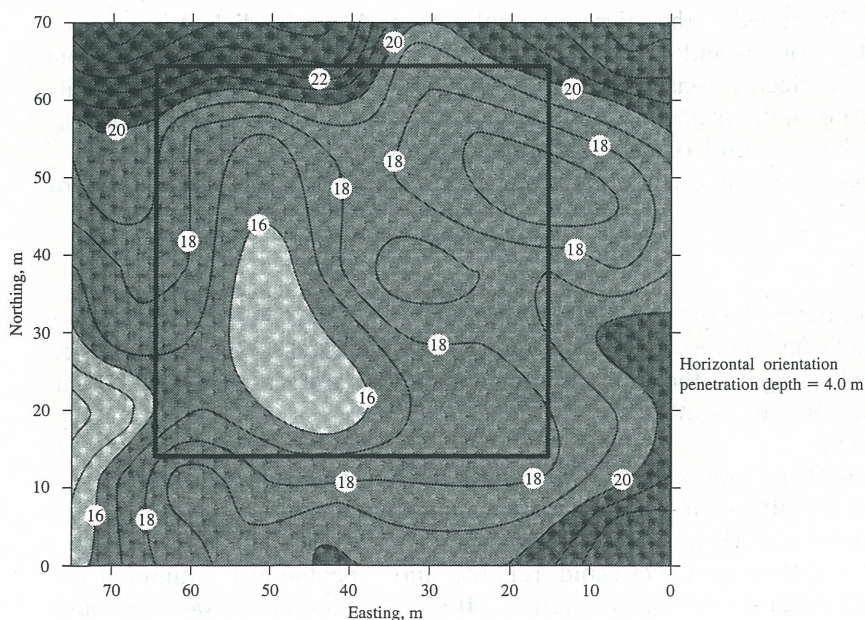

Figure A1.2   Contour map showing horizontal field response of EM-38 manufactured by Geonics Ltd. Data from Young et al. (1996)

inside a pvc source tube, usually through a series of copper plates, at specific depth intervals, to a detecting electrode some distance away. The detectors are usually also copper plates attached to a detection device placed within a similar tube. The detector measures the electrical resistivity of the soil, which is the inverse of electrical conductivity. Measurement is a function of both ionic and water content. Resolution of data is proportional to detection-well spacing; the lower the soil resistivity, the closer the wells must be. A commonly used depth is 1.5 times the distance between wells. For example, wells 10 m apart should be about 15 m deep. The data obtained is sorted via a modeling program and yields a three-dimensional plot of the wetting front. This procedure is valuable for long-term monitoring strategies, but is expensive. It is currently in use by the United States Nuclear Regulatory Commission on its Maricopa, Arizona field-studies evaluation site.

Suitability Criteria: **(1)** possible to merge with other devices; **(2)** durable (however, there is limited experience with the new vertical installation and measurement procedure); **(3)** flexible; and **(4)** service not required.

Advantages and Disadvantages: **(1)** measurements are large scale; **(2)** valuable for long-term studies such as landfills and low-level radioactive waste disposal sites; **(3)** detection wells should be installed during initial site installation; **(4)** measurements are not intrusive; **(5)** maintenance not required on soil probes; **(6)** cross-hole measurements must have sophisticated equipment and data analysis; **(7)** expensive (approximately $10,000); **(8)** requires on-site AC power; and **(9)** commercial units are not readily available.

Matric Potential Measuring Devices

Tensiometers   Tensiometers measure the energy status of water in the soil matrix (i.e., matrix potential). A tensiometer consists of a liquid-filled, unglazed porous ceramic cup connected to a pressure measuring device (such as a vacuum gauge or transducer). Once the ceramic cup is embedded into soil, soil solution can flow into or out of the cup via small pores in the ceramic. The flow will continue until the pressure potential of the liquid inside the cup equals the pressure potential of the soil water around the cup. If the column is completely filled with liquid, the matric potential will be zero or near zero, and no solution will flow into or out of the cup. As the soil dries, the solution will flow out of the cup and the top of the column will recede, creating an air pocket near the pressure measuring device and, thus, a vacuum. The vacuum created will be measured as a pressure or negative suction by the vacuum gauge or transducer. Examples of tensiometers used by the authors are illustrated in figure A1.3.

Suitability Criteria: **(1)** possible to automate and merge with other devices; **(2)** durable, but requires maintenance; **(3)** less flexible installation; **(4)** "Sisson" type tensiometer can be

Small porous cup design
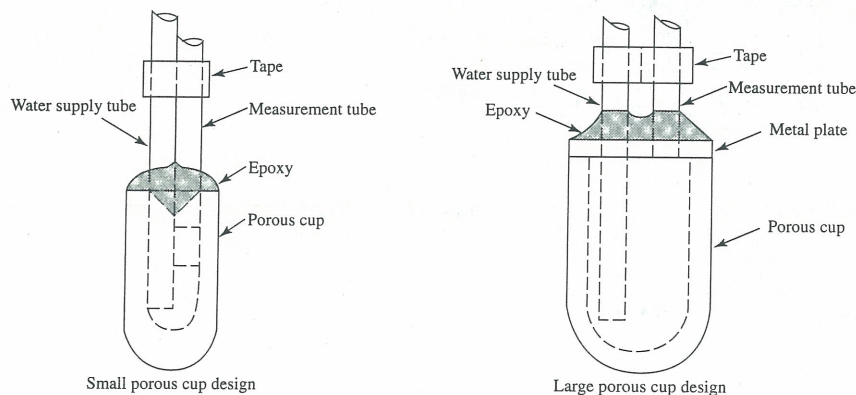
Large porous cup design

Figure A1.3   Tensiometer design. Small porous cup diameter (outside) is approximately 1 cm. Data from Stannard (1992)

easily serviced and overcomes the depth limitation (less than 2–5 m) of most tensiometers. The Sisson tensiometer is referred to as the "advanced tensiometer," and was developed by personnel at the Idaho National Engineering Laboratory (INEEL). The Sisson design is shown in figure A1.4.

Advantages and Disadvantages: **(1)** works only in the moist range (0 to −85 kPa; 100 kPa = 1 bar); **(2)** nominal cost (approximately $37.00 each, regardless of type); and **(3)** frequent maintenance required.

Heat dissipation sensor (HDS)   An HDS measures thermal diffusivity by applying a heat pulse to a heater located within the ceramic cup, then monitoring the temperature in the center of the cup before and after heating. The measurement system is generally a diode bridge circuit that measures electromotive force generated by the change in diode temperature in the sensing element as the heat pulse is applied. The higher the water content, the greater the thermal conductivity and diffusivity of the soils and hence, the lower the measured electromotive force. As the soil drains, causing the ceramic to desorb, thermal conductivity and diffusivity decrease, causing an increase in temperature in the reference matrix material. An HDS is illustrated in figure A1.5.

Suitability Criteria: **(1)** possible to automate and merge into a monitoring system; **(2)** durable; **(3)** flexible installation; and **(4)** not easy to remove or service.

Outer casing:
1.0″ ID Class 200 PVC

Sieved native material—backfilled dry

Inner casing:
0.5″ ID Schedule 40 PVC, or equivalent

Pressure transducer—two-pair wire extended to ground surface

One-hole rubber stopper

Porous cup—1 bar, high flow

Sieved native material—backfilled in heavy slurry
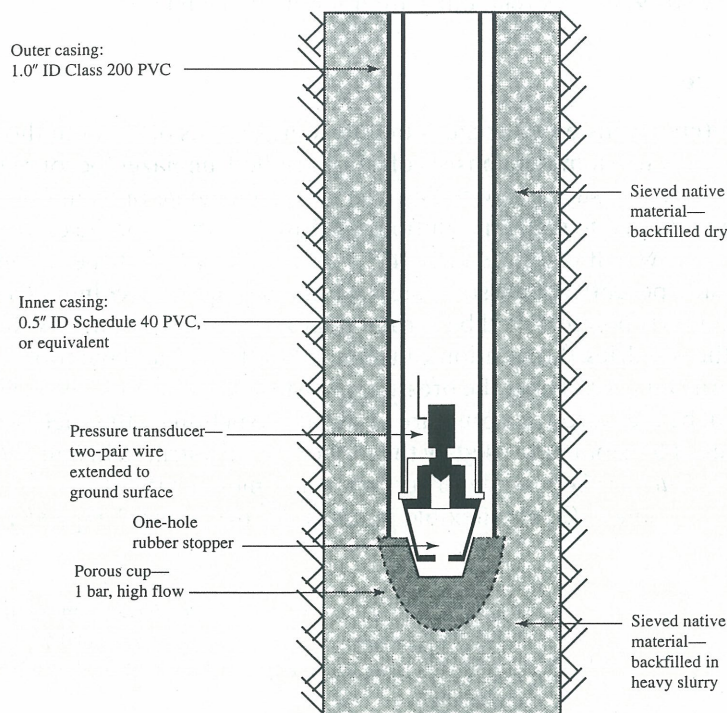
Figure A1.4   Expanded view of "advanced tensiometer" system. Data from Young et al. (1996)

Porous ceramic plug

Line heat source and thermocouple

1.5 cm

3.0 cm

Thermocouple and heat source wire leads

Figure A1.5   Heat dissipation sensor

Advantages and Disadvantages: **(1)** point measurement; **(2)** relatively new, less than five years experience; and **(3)** wide range (10–1500 kPa)—however, some scientists report poorer results in wetter soils (0–0.5 kPa).

Thermocouple psychrometers   These devices are discussed in chapter 9.

Suitability Criteria: **(1)** possible to automate and merge into a monitoring system; **(2)** variable durability with varying soil conditions; **(3)** flexible installation; and **(4)** not easy to remove or service.

Advantages and Disadvantages: **(1)** complex measurement; **(2)** low durability; **(3)** point measurement; and **(4)** only operable in dry soils, from 50–3000 kPa.

Soil Solution Sampling Devices

Solution samplers   A solution sampler obtains a soil water sample through a porous wall. The common name for this device is "suction lysimeter." The solution sampler has two tubes that enter the device; one is an air pressure or vacuum tube, the other a fluid return tube. Once installed, a solution sample is easily obtained. A vacuum equivalent to soil pressure is applied via the vacuum tube, and both tubes are clamped or pinched shut. Once the pressure is equilibrated within the sampler, the clamps on the tubes are removed, a sample bottle is attached to the fluid return tube, and a pressure exerted on the vacuum tube to push the sample from the ceramic cup into the sample bottle. These devices come in both single- and dual-chamber designs, and are made from either ceramic or stainless steel. A dual-chamber, stainless-steel solution sampler is shown in figure A1.6.

Figure A1.6   Diagram of dual-chamber solution sampler. Data from Wierenga et al. (1993)

Suitability Criteria: **(1)** possible to merge into a monitoring system; **(2)** durable; **(3)** limited installation flexibility; and **(4)** not easy to remove and service.
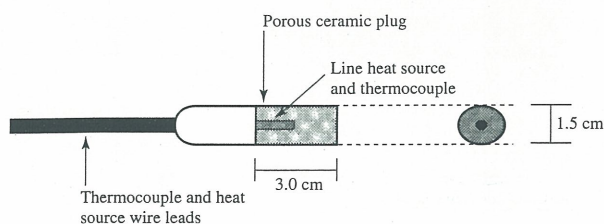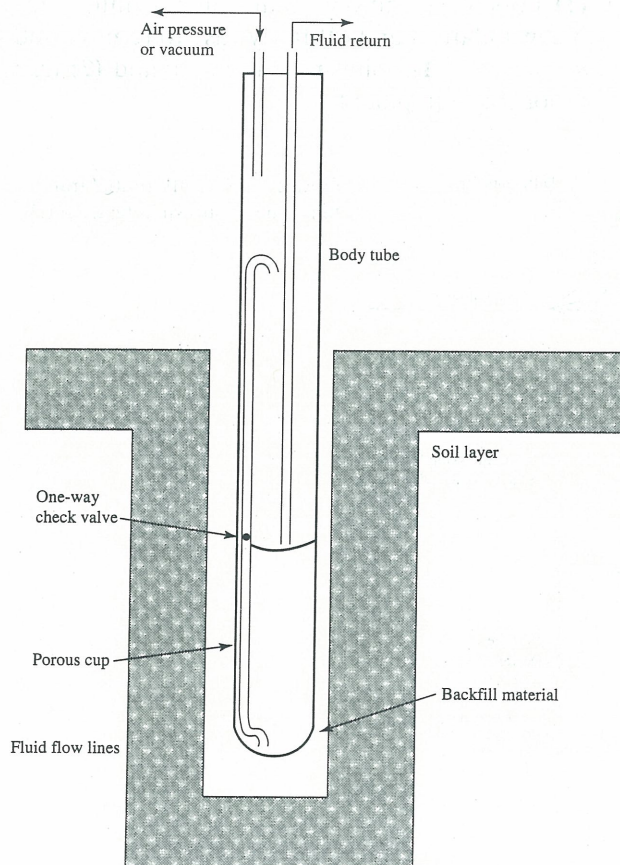
Advantages and Disadvantages: **(1)** not readily automated; **(2)** small capture area; **(3)** limited to shallow depths, usually less than 10 m; and **(4)** limited to soil water contents characteristic of 0–50 kPa pressure.

Sampling Wells   Sampling wells for saturated zone work may be necessary. However, many other texts present detailed information on this subject. Therefore, the reader is referred to Freeze and Cherry (1979).

Suitability Criteria: **(1)** not easily automated, but possible to merge into a monitoring system; **(2)** durable; **(3)** flexible installation; and **(4)** not easy to remove and service.

Advantages and Disadvantages: **(1)** point measurement; and **(2)** very repeatable.

Pressure Measurement Devices: Differential Pressure Transducers

There are two primary devices used for measuring soil pressure, vacuum gauges and differential pressure transducers. Commonly, each device is attached atop a liquid-filled column, which is in turn attached to a porous cup. Because vacuum gauges are relatively older technology, they shall not be discussed.

Differential pressure transducers are usually four-active-element piezoresistive bridge devices. When installed, and a pressure is applied, a differential output voltage proportional to that pressure is produced. This output voltage is commonly measured with a data logger. A differential transducer connection to the tensiometers illustrated in figure A1.3 is shown in figure A1.7.

Suitability Criteria: **(1)** possible to automate and merge into a monitoring system; **(2)** fairly durable; **(3)** flexible installation; and **(4)** fairly easy to remove and service.

Advantages and Disadvantages: **(1)** point measurement; and **(2)** transducers are delicate and require use of data-logging equipment.
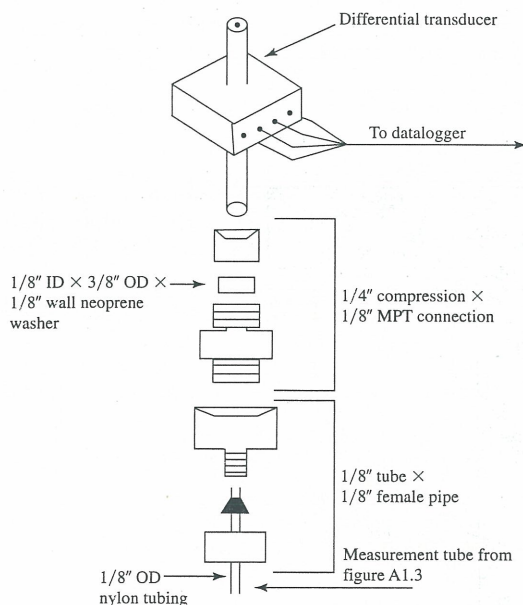

Figure A1.7   Differential transducer attachment to tensiometer shown in figure A1.3

APPENDIX 2

Mathematics Review

INTRODUCTION

Over the years, we have noticed that an increasing number of students taking our unsaturated zone hydrology courses have come from a broadening variety of scientific disciplines. Many of these students attend our classes to expand their skills in environmental science. Some have been out of school for several years, or are retraining in another discipline. A number of these students have a minimal mathematics background. Consequently, we have experienced a need to review basic math skills, so students can adequately understand the material presented in the text. For those readers who need a refresher in mathematics, we hope that this appendix will serve that purpose.

ALGEBRA

The basic rules for the arithmetical manipulation of a number system are known as algebra. They consist of two operations, addition ($+$) and multiplication ($\cdot$), and eleven laws relating those operations to the number system. For any a, b, or c within a number system, the following properties hold.

1. $a + b = b + a$ *Commutative Law of Addition*
2. $a \cdot b = b \cdot a$ *Commutative Law of Multiplication*
3. $a + (b + c) = (a + b) + c$ *Associative Law of Addition*
4. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ *Associative Law of Multiplication*
5. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ *Distributive Law of Multiplication over Addition*
6. There exists a number a such that $a + b = b$ (and $a = 0$). *Existence of 0, the additive identity*
7. There exists a number a such that $a \cdot b = b \cdot a = b$ (and $a = 1$). *Existence of 1, the multiplicative identity*
8. For any number a, there exists a number b, for a and $b \neq 0$, such that $a \cdot b = 1$ (and $a = 1/b$). *Existence of reciprocals*
9. For any number a, there exists a number b such that $a + b = 0$ (and $b = -a$). *Existence of negative numbers*
10. If a is in the number system, and b is in the number system, then $a + b$ is in the number system. *Closure under addition*
11. If a is in the number system, and b is in the number system, then $a \cdot b$ is in the number system. *Closure under multiplication*

These laws enable one to structure a problem in such a way that it can be solved quickly and consistently. For example, $4 \div 7 \div 3$ can have two different answers depending upon the order in which it is solved, since division is not associative. However, if the problem is stated as $4 \cdot (1/7) \cdot (1/3)$, the answer remains the same no matter what order one chooses to solve it. Likewise, $4 - 7 - 3$ can have two different answers depending upon the order in which it is solved, whereas $4 + (-7) + (-3)$ is consistent. It is these laws of algebra that dictate how equations are set up and solved in the most advanced forms of mathematics.

The smallest algebra possible consists of two numbers $(0,1)$ arranged in a matrix to meet the eleven stated requirements. However, that algebra is of little or no consequence to the applied sciences. The next-smallest number set that meets those requirements is the set of rational numbers. Unfortunately, that set does not contain such important numbers as π or e, or even $\sqrt{2}$; but those numbers *are* contained in the real number system, which is the number system that will be used as a basis for this review. The complex number system, which consists of the real and imaginary numbers (imaginary numbers are those that must be described in terms of i, or $\sqrt{-1}$), also forms an algebra. Although the complex numbers are of considerable importance in fields such as electrical engineering, the real number system is the one that is of primary importance in most applications.

SOME BASIC DEFINITIONS

Since mathematics is the language of numbers, it might help to review some of the basic definitions.

variable   A variable, which is usually represented by a letter of the Roman alphabet, can take on more than one value; or it may represent a single value that is yet to be determined. In the equation $3x + 2 = y$, x is the *independent* variable, and y is the *dependent* variable.

operator   A symbol that is used to denote a mathematical manipulation (operation). In essence, operators are mathematical shorthand: $+, -, \times, \div, \sqrt{\ }, \int, \nabla$, and Σ are all examples of operators. Likewise, the use of a superscript is a common operator. When one sees x^7, one knows that it is shorthand for the operation $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$.

function   A combination of operators, numbers, and/or variables that lead to a single value. In many cases, the exact mathematical expression may not be given, because it varies considerably or simply is not known. $f(x)$ indicates that the function f involves operations on the single variable x. $G(x, y, z)$ indicates that the function G involves operations on three variables. Neither of these examples gives an indication of the operations to be performed on the variables, only the number of variables involved. However, each function will have only one value for each set of variables. For example, if $G(1, 3, 2)$ is a function, it takes only one value for that number sequence $(1, 3, 2)$. It may have the same value for some other set of numbers, but for the sequence $(1, 3, 2)$ it must always take the same value. It cannot be 7 on Wednesdays and 12 on Fridays; once a value is assigned to a sequence of variables in a function it must remain constant for that sequence of variables. That strictness of definition is not always recognized in the physical sciences, but is dealt with by using a little common sense. As an example, consider Kepler's third law of planetary motion. The law can be stated mathematically as $T^2 = (4\pi^2 a^3 / GM)$; where T is the time for the period of the planetary orbit, a is the mean orbital radius, M is the mass of the planet, and G is the universal gravitational constant. By algebraic manipulation, the equation can be expressed in

such a way as to show any of the variables as the dependent variable:

$$T = \left(\frac{4\pi^2 a^3}{GM}\right)^{1/2}; \quad a = \left(\frac{T^2 GM}{4\pi^2}\right)^{1/3}; \quad M = \left(\frac{4\pi^2 a^3}{GT^2}\right) \quad \text{(A2.1)}$$

Of the three equations given, two are functions and the other is not! T can have two possible values for any set of values for the independent variables a and M: the negative and the positive square roots of the equation. However, if the problem is set up with the understanding that T can only take positive values, the equation can become a function. $T(M, a)$ is a function for $T > 0$.

monomial	A monomial in x is of the form cx^n, where n is a nonnegative integer and c is a constant; for example, $-3x^5, 4x, 7x^{12}$.
polynomial	A series of monomials added together is a polynomial; for example, $7x^{12} - 3x^5 + 4x$.
linear polynomial	A polynomial is linear if it has the form $ax + b$, where $a \neq 0$.
quadratic polynomial	A polynomial is quadratic if it has the form $ax^2 + bx + c$, where $a \neq 0$. The roots of a quadratic polynomial can be found by using the equation $[-b \pm (b^2 - 4ac)^{1/2}]/2a$.
cubic polynomial	A polynomial is cubic if it has the form $ax^3 + bx^2 + cx + d$, where $a \neq 0$.

Equations are often plotted using the Cartesian coordinate system. When a linear polynomial of the form $y = mx + b$ is plotted with Cartesian coordinates, m is the slope (tangent) of the line, and b is the y-intercept (the value of y where the line crosses the y-axis). Likewise, for any two points (x_1, y_1), (x_2, y_2), the slope of the line between them is $m = (y_2 - y_1)/(x_2 - x_1) = \Delta y/\Delta x$. Changes in b only move the line up or down the y-axis.

Although this definition of a tangent applies to straight lines, and the tangent of a curve can be found through differential calculus, there are often circumstances where the tangent to a curve can be approximated by a line. For that reason, it helps to understand calculus; one can approximate the tangent of a line at just a glance. It also helps one to recognize errors in calculation. Figure A2.1 gives an idea of the ranges of values for the tangents of lines.
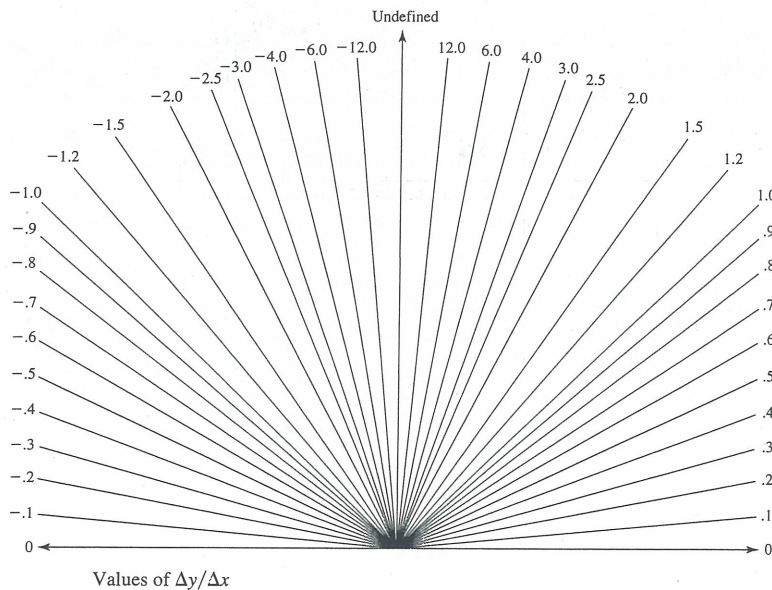

Figure A2.1 Tangent lines of the Cartesian coordinate system

Values of $\Delta y/\Delta x$

TRIGONOMETRIC RELATIONS

The following discussion utilizes the diagram in figure A2.2. The radius of the circle, r, is 100 mm; the area of the circle is πr^2; the circumference of the circle is $2\pi r$. Angles can be measured in either degrees or radians, and there are 360°, or 2π radians, in a complete circle. Let α_r = angle measured in radians, and α_o = angle measured in degrees. The relation can be described by: $\alpha°/360° = \alpha_r/2\pi$. This relation allows one to convert degrees to radians, or radians to degrees, for any angle. In figure A2.2, $\alpha°/360° \cdot 2\pi = \alpha_r$, and $\alpha_r \cdot r = z$. The area of the wedge is $\frac{1}{2}\alpha_r r^2 = \frac{1}{2}z \cdot r$. It can readily be seen that angle measurements in radians are important in measuring areas, and areas of portions of circles.

Trigonometric Functions

Considering the triangle formed by the radius r, height y, and base of length x, the trigonometric functions are as follows.

$\sin \alpha = y/r$

$\cos \alpha = x/r$

$\tan \alpha = \sin \alpha/\cos \alpha = y/x$, undefined for vertical line, $\alpha = 90°, \pi/2$ rads

$\csc \alpha = 1/\sin \alpha = r/y$, undefined for horizontal line

$\sec \alpha = 1/\cos \alpha = r/x$, undefined for vertical line

$\cot \alpha = 1/\tan \alpha = x/y$, undefined for horizontal line

The area of the triangle is $\frac{1}{2}xy$.
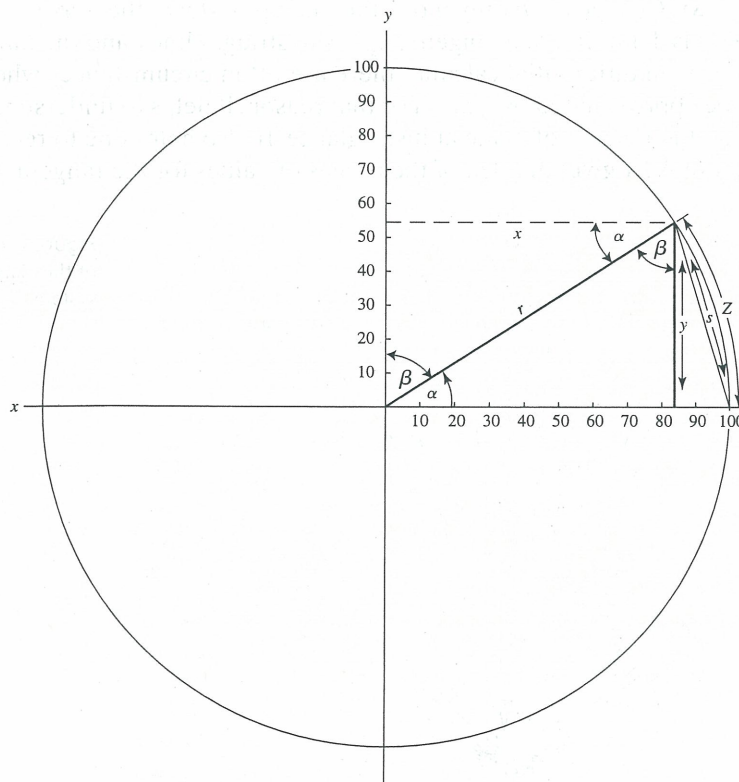
Pythagorean theorem: $x^2 + y^2 = r^2$.

Figure A2.2 Relation of trigonometric shapes

The length of the chord (secant segment) labeled s can be found by using the law of cosines: $s^2 = 2r^2 - 2r^2 \cos \alpha \to s^2 = 2r^2(1 - \cos \alpha) \to s = r(2(1 - \cos \alpha))^{1/2}$ for $s \geq 0$. The area of the triangle formed by the chord and the two radii is $\frac{1}{2} yr$.

The area within the arc and chord is $\frac{1}{2} zr - \frac{1}{2} yr = \frac{1}{2} r(z - y)$.

Important Note: If any value for an angle is not specifically given in degrees, it is assumed to be in radians. For example, sin 30 is assumed to be sin 30 rads, not sin 30°!

The following laws and identities are basic, and are not restricted to figure A2.2.

Law of Cosines

For any triangle with sides of length a, b, and c, if θ represents the angle opposite side c, then $c^2 = a^2 + b^2 - 2ab \cos \theta$.

Law of Sines

For any triangle with sides of lengths a, b, and c, if θ represents the angle opposite c, α represents the angle opposite a, and β represents the angle opposite b, then $a/\sin \alpha = b/\sin \beta = c/\sin \theta$.

For any α, β, θ (except as noted), the following hold.

$\sin^2 \alpha + \cos^2 \alpha = 1$

$\sin (-\theta) = -\sin \theta$

$\cos (-\theta) = \cos \theta$

$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

By substituting 90° ($\pi/2$ radians) for α in the preceding four equations, we obtain the following.

$\sin (90° - \beta) = \cos \beta$

$\cos (90° - \beta) = \sin \beta$

$\sin (90° + \beta) = \cos \beta$

$\cos (90° + \beta) = -\sin \beta$

If $\alpha = \beta$ in those same four equations, then the following is true.

$\sin (2\alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha = 2\sin \alpha \cos \alpha$

$\cos (2\alpha) = \cos^2 \alpha - \sin^2 \alpha$

Dividing $\sin^2 \theta + \cos^2 \theta = 1$ by $\sin^2 \theta \to 1 + \cot^2 \theta \csc^2 \theta$

Dividing $\sin^2 \theta + \cos^2 \theta = 1$ by $\cos^2 \theta \to \tan^2 \theta + 1 = \sec^2 \theta$

All of the useful trigonometric identities can be derived by similar substitutions, but these are the most important.

GEOMETRIC RELATIONS

The *area of a triangle* is $\frac{1}{2} bh$, where b is the length of the base and h is the triangle's height. Please note that h is the same length as the triangle's leg only in the case of a right triangle.

A *right triangle* is one in which one of the angles is equal to 90°.

The total inside angles of a triangle sum up to 180°.

An *isosceles triangle* is one in which two of the sides are equal. Note that by the law of sines, that means that two angles are equal also.

An *equilateral triangle* is one in which all three sides are equal. Once again, by the law of sines all three angles must also be equal, and therefore each is 60°.

The *total area of a polygon*, as well as the measurements of its inside angles, can be determined by constructing a series of triangles within the polygon and using the above relations.

A *parallelogram* is a four-sided polygon with its opposing sides parallel and equal in length. It can be formed by two identical triangles, therefore its area is $2(\frac{1}{2}bh) = bh$. Its inside angles total $2 \cdot 180° = 360°$. Squares and rectangles are special cases of parallelograms.

An *ellipse* has the algebraic form $x^2/a^2 + y^2/b^2 = 1$; where a is the length from the center of the ellipse to its furthest point (also known as the "semi-major axis"), and b is the length from the center of the ellipse to the closest point (also known as the "semi-minor axis"). (This definition assumes that the widest portion of the ellipse coincides with the x axis, which is conventional.) It can also be seen that when $a^2 = b^2$, the equation of an ellipse reduces to $x^2 + y^2 = a^2$, which is the equation of a *circle;* showing that the circle is a special case of an ellipse. The geometric definition of an ellipse is that it is a set of points such that the sum of the distances of each point from two fixed points is a nonnegative constant. The two fixed points are known as the foci. It is that definition that leads to the traditional means of constructing an ellipse by using two thumbtacks, a length of string and a pencil. One focus lies on each semi-major axis, and each is the same distance from the center of the ellipse. If the distance of each focus from the center is given as c, then the axes and foci are related by the following equation: $b^2 + c^2 = a^2$. Once again, in a circle, c reduces to zero since the foci coincide at the center. A line that is tangent to an ellipse at any point will form the same angle to a line drawn to either of the two foci. The area of an ellipse is πab.

Volume of an elliptic cylinder $= \pi abh$ (where h is the height)

Volume of a cone $= \frac{1}{3} \pi r^2 h$.

Surface area of a cone $= \pi rl$ (where l is the distance along the edge of the cone from its apex to its base).

Volume of a sphere $= \frac{4}{3} \pi r^3$.

Surface area of a sphere $= 4 \pi r^2$.

POWERS AND ROOTS

One area of mathematics that is actually algebra, but is difficult enough to cause problems even for those who may have taken calculus courses, is the concept of *powers* and *roots*. The following rules are basic to an understanding of the subject.

$x^0 = 1$

$x^1 = x$

$x^2 = x \cdot x$

$x^n = x_1 \cdot x_2 \cdot \cdots \cdot x_n$

Let $\sqrt[n]{a} = x$, then $x^n = a = x_1 \cdot x_2 \cdot \cdots \cdot x_n$, and $x = a^{1/n}$

$x^{-n} = 1/x^n$

Let $r = x^m$ $s = x^n$, then $r \cdot s = x^{m+n}$

$(x^m)^n = x^{m \cdot n}$

For any $y > 0, x > 0$, there is some number n such that $x^n = y$. This fact leads to the concept of logarithms.

LOGARITHMS

For all numbers greater than zero, logarithms provide a simplified way to deal with powers and roots. Logarithms are expressed in terms of a base. In the example given in the last sentence of the previous paragraph, n would be the logarithm of y in base x. The mathematical expression would be $\log_x y = n$. The two most common bases for logarithms are 10 and e. If a number is expressed as $\log y$, it is understood that it is meant as $\log_{10} y$. If a number is expressed as $\ln y$, it is understood that it is meant as $\log_e y$. Therefore, $\log y = n$ means that $10^n = y$, $\ln y = n$ means that $e^n = y$. Some of the rules that apply to logarithms are as follows.

$$\log_x x = 1$$
$$\log_x x^a = a$$
$$\log xy = \log x + \log y$$
$$\log x/y = \log x - \log y$$

The relation between ln and log ($\log_{10}$) is as follows

$$\ln 10 = 2.30258\ldots$$
$$e^{2.30258} = 10$$
$$e = 10^{0.434294\ldots} \text{ (since } 1/2.30258\ldots = 0.434294\ldots\text{)}$$
$$\log e = 0.434294\ldots$$

For any $x > 0$, $\ln x = 2.30258 \log x$.

This relation is extremely important in the physical sciences. When a differential equation is solved, the answer is often expressed in terms of ln, but experimental data is more conveniently plotted on $\log_{10}$ graphs.

CALCULUS

Differential and Integral Calculus

There are two essential areas of calculus. One is known as *differential calculus,* and is concerned with finding the tangent to a curve at a particular point. The other is known as *integral calculus* and is concerned with finding the area beneath a curve between two points. Although this is a simplification in two dimensions, if one remembers to look at calculus in this simple way it can help to make some problems very easy to solve. That is especially true in the area of applied calculus.

As stated above, differential calculus is concerned with finding the tangent to a curve at a point. The slope of a tangent line at a particular point on a curve is the instantaneous rate of change. The slope of that line is known as the derivative of the equation. The mathematical representation for that instantaneous rate of change, or derivative, is dy/dx. Remember that $\Delta y/\Delta x$ is the representation for the slope of a straight line. There are times when a curve does not change dramatically over a distance; in such an instance dy/dx may actually approach $\Delta y/\Delta x$, making it easier to approximate dy/dx without knowing the actual formula of the curve. That is very important to a scientist, since the actual formula for an experimentally-derived curve may be solvable only through numerical analysis of data. There can be many instances where a scientist may want approximate values before spending long hours developing a computer program to analyze data. A scientist may also desire to assess the data presented by someone else without spending huge amounts of time.

The integral for a curve between the points a and b, which is the area within (beneath) the curve between those points, is represented by $\int_a^b$. In the Cartesian coordinate system, it would be represented by $\int_a^b$ (equation of curve) dx. The representation that shows the two points between which the curve is being evaluated is called the definite integral. The

indefinite integral $\int$ (equation) dx, is also called the antiderivative. That is because it undoes the effect of the derivative. The integral of a derivative does not always yield the original equation completely, but it is close enough for the purposes of calculus. It is not the intent of this discussion to go into an explanation of integrals or derivatives, so if a more rigorous explanation or review is needed there are literally scores of calculus books available in any library. The important concept here is that a definite integral represents the area beneath two points on a curve, and the antiderivative "undoes" the effect of the derivative.

If one knows the exact equation of a function, the finding of a derivative at a particular point is a mathematical exercise in differential calculus. However, the purpose of performing experiments is to find unknown values. An experiment may obtain a whole series of values that can be linked together in a curve sketched by the researcher, but unless the curve is very simple it may not be easy to identify its underlying mathematical equation (function). What is a researcher to do in order to find the equation, or its derivative at a certain point? A French engineer named Fourier established that a curve (or even a series of segments of curves) of any shape can be matched against segments of trigonometric functions (sine or cosine), as long as the curve is a function. But Fourier analysis is not a simple exercise to be undertaken with a notepad, pencil, and hand calculator. If all a scientist needed was a value at a particular point, it could be read from the curve and no further calculations would be needed. Yet, quite often the scientist needs to find the derivative at a point, or the derivatives at several points. When a rough approximation is adequate, dy/dx can be obtained by using the chart given in figure A2.1 and finding the value of $\Delta y/\Delta x$ that most closely matches the slope of the curve in the immediate vicinity of the point.

The value of the definite integral can be approximated by plotting the curve of experimental data on quadrille paper and determining the number of squares contained under the curve between the two points. In calculus texts one might find such methods of approximation listed under the category of "trapezoidal approximation."

Consider the task of designing a mathematical function that will describe a trip from the Colorado capitol building in Denver to the summit of Mount Evans (figure A2.3). The capitol will serve as a point of origin, East–West will be the x-axis and geographic North–South the y-axis. It will be impossible to describe the trip as a function in terms of either x or y alone, because the road loops around with various switchbacks as one travels in the mountains; as a consequence East–West or North–South lines can be drawn in such a way that they can pass over the road several times. One may be tempted to make the assumption that the trip can be described in terms of x and y, or $f(x, y)$, as the exact mathematical equation is unknown. But consider what happens when one encounters a cloverleaf intersection. The vehicle turns steeply to the right and within a few seconds one is at precisely the same coordinates as a few seconds before—but not in the same place. The vehicle is approximately eighteen feet above where it was. The contemplated function $f(x, y)$ gives two values at one point; therefore, it cannot be a function after all. In order to meet that difficulty, the function can be changed to $f(x, y, z)$, with z the elevation above, or below, that at the capitol. One still does not know the exact mathematical equation for the journey, but a specific set of values (x_1, y_1, z_1) will give just one location, so there is a function $f(x, y, z)$. For any set of values, one can find the exact position on a topographic map. The topographic map is a two-dimensional representation of that three-dimensional function. The mathematical equation will be tremendously involved and very difficult to determine. Now consider the following philosophical question: Do you really need to know the exact mathematical expression in order to understand the function, or do you actually understand the function better without knowing the exact mathematical expression?

Next consider what is involved in the trip back down from the summit of Mt. Evans. Suppose that one lane has been closed for construction so that the trip back down exactly retraces the route up for at least part of the way. Given a set of coordinates (x_1, y_1, z_1) that

★
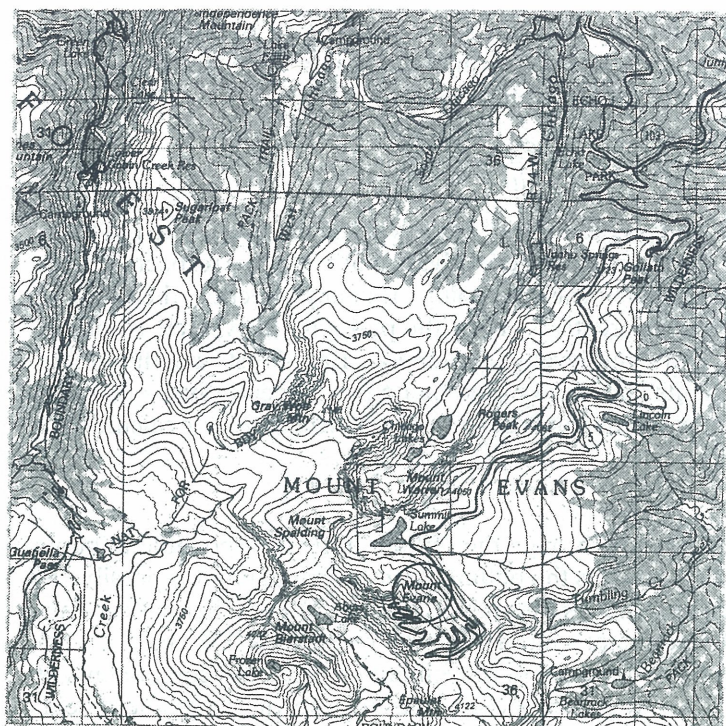
Denver

Figure A2.3   Mount Evans, 45 km west of Denver, Colorado (not to scale). Draw imaginary x- and y-axes through highway 5 in proximity of the summit of Mount Evans. If the y-axis is drawn in the appropriate location, one will observe that for each x, there can be multiple locations along y, and vice versa. Map scale is 1:100,000. Source: United States Geological Survey. Map name: Denver West—Colorado

falls somewhere on that section of road; what does $f(x_1, y_1, z_1)$ give for an answer? Is the vehicle on its way to the summit or on its way back down at that point?

There might be a temptation to suggest that time is another variable. But then what happens if the occupants of the vehicle stop to admire the view, or are held up in the construction zone? Or what if someone left a pair of binoculars behind, and the vehicle turns around and heads back up again? If someone told you that it was precisely three hours, twenty-seven minutes and fourteen seconds since the vehicle left the capitol, could you give its exact location? It is obvious that time alone is not what is needed. Reconsider for a moment the question that must be resolved. One only needs to determine whether the vehicle is moving in the direction toward the summit or away from it. Movement implies change in position with time. This is the same as finding a derivative in calculus. In a one-dimensional problem along the x-axis, this would be solved by finding the derivative dx/dy or dx/df; but this is a three-dimensional problem. You might find a short distance where the road is completely level and heads precisely North–South or East–West, but at most times it will not. Consider for a moment what happens when there is vertical change. Unless the driver propels the vehicle over the edge of a cliff there will always be a considerable rate of change along at least one of the other major axes. The variables are interlinked in the underlying function. Change cannot occur in z without change also occurring in x or y (in this case). Even in the x–y plane there are many sharp curves and changes in direction, so their relation is much more involved than that of a straight line, $mx + b = y$. Even if the vehicle were to move at a constant velocity relative to the road surface, its rate of change with respect to any of the three axes would change constantly. If one considered the rates of change along each of the three axes at a particular point, while keeping each of the other variables a constant, one would have found the partial derivatives:

$$\frac{\partial f}{\partial x}, \qquad \frac{\partial f}{\partial y}, \qquad \frac{\partial f}{\partial z} \tag{A2.2}$$

The tangent to the curve of the road at any point is the set of partial derivatives at that point. In the example given above, if time enters into the format, the relation between the regular derivatives and the partial derivatives would be as follows.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \tag{A2.3}$$

The instantaneous rate of change, df/dt, is the velocity of the vehicle relative to the road surface, and dx/dt, dy/dt, and dz/dt are the regular derivatives relative to the respective axes. The velocity has components that could be measured going off in any direction; the function given above shows where the velocity is greatest—along the path that the vehicle is following. If the vehicle were to follow a straight, level path along an East–West line, then there would cease to be y and z components and there would be no partial derivatives, and df/dt would be equal to dx/dt. This lengthy discussion has led to the subject of vector calculus.

Vector Calculus

Consider for a moment a pollutant moving through the soil or air. It is not possible to know the exact mathematical equation that governs the pollutant's movement, but it is possible to measure its movement at any one point (i.e., as through Δx in figure A2.4). There are an infinite number of points at which the pollutant's movement can be measured. At each one of these points the partial derivatives (the direction of movement measured as components of the three axes) may be different. Taken together, these points are known as a gradient field. The set of partial derivatives at any given point is known as the gradient. The mathematical symbol for the gradient is grad $f(x, y, z)$, or more commonly as $\nabla f(x, y, z)$. It is vital to remember that it is possible to know by experimental measurement the value of $\nabla f(x, y, z)$ at a point without knowing anything at all about the underlying function $f(x, y, z)$.

Nabla (in older usage; del, or del operator in newer usage), symbolized by ∇, is the most important operator in applied vector calculus. $\nabla f(x, y, z) = $ grad $f(x, y, z)$ is the gradient, or the set of partial derivatives at a point. $\nabla \cdot f(x, y, z) = $ div $f(x, y, z)$ is the divergence. ($\nabla \cdot$ is read as "nabla dot"). $\nabla \times f(x, y, z) = $ curl $f(x, y, z)$ is the curl. ($\nabla \times$ is read as "nabla cross"). An extremely simplified explanation of this notation is: $\nabla f(x, y, z)$, which is the tangent to a gradient field at a particular point (the direction of the maximum rate of change broken into its three components). $\nabla \cdot f(x, y, z)$ is the normal to the gradient field (the rate of change 90°
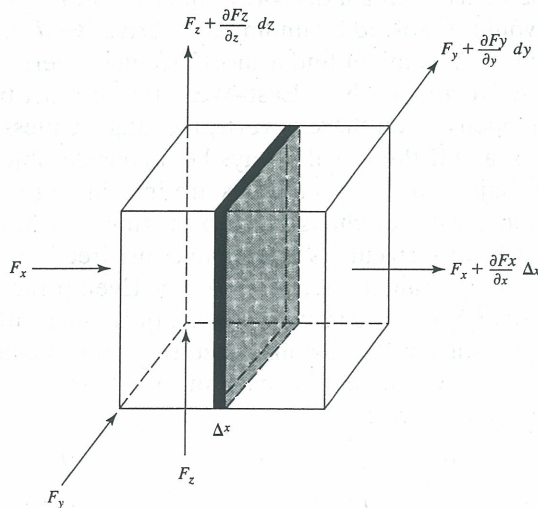

Figure A2.4   Flux across an interval, Δx, within a three-dimensional box

to the gradient field, or the expansion of a field at a particular point). Once again, it would be broken into the three components. If the field is confined within boundaries, then $\nabla \cdot f(x, y, z) \equiv 0$. $\nabla \times f(x, y, z)$ can be considered to be the measure of rotational motion about a point in the gradient field. (Another important, but complicated, use of ∇ is $\nabla^2 f(x, y, z)$, the LaPlacian of a function. The LaPlacian combines the important mathematics of differential equations and vector calculus.)

The actual calculation of these vector quantities using the underlying mathematics would be very difficult. It is very fortunate that the quantities can be measured without even knowing the underlying equation. This very short discussion of vector calculus should enable you to understand what would otherwise be very intimidating mathematical formulas that you will come across. A short example follows. Take a look at it, and see if you can interpret it by yourself.

$$\frac{\partial \theta_v}{\partial t} = \nabla \cdot (D_{\theta v} \nabla \theta_i) + \nabla \cdot (D_{Tv} \nabla T) + E \tag{A2.4}$$

This equation simply says that the partial derivative with respect to the variable time (you will note that it is not concerned with any of the other partials) in the function θ_v is the sum of the other three entries. The two $\nabla \cdot$ components are the expansion of the formula contained within the brackets. Although we do not have the benefit of knowing what each variable stands for, the way that they are being used, as well as their being upper case letters, would seem to indicate that each letter represents some equation that is unknown to the person who created the formula. Therefore, each of these components represents an experimentally derived number that the individual believes to be a function, as indicated by the use of capital letters. ∇T would represent the temperature gradient, a number that should be easily obtained. Likewise, $\nabla \theta$ represents some measurable gradient. Even without knowing what each of the variables means, the overall equation is going to be easily solved. Once the nature of the function is understood, an experiment can be designed to derive an answer, and you will not have to be an expert mathematician in order to solve it!

Differential Equations

One of the most common experimentally obtained results is known as the ordinary differential equation. A differential equation occurs when a measured rate of change is equal to a function of its variables: $dy/dx = f(x, y)$ is an example of how this might be represented in mathematical notation. Differential equations can get very complicated, so this discussion will center on the easiest and (fortunately) most common type, the form $dy/dx = ky$, where k is some constant. By separating the variables one obtains $dy/y = kdx$. Taking the antiderivatives of each side, $\int dy/y = \int kdx \rightarrow \ln |y| = kx + c$. This can also be expressed as $y = e^{kx+c}$, or $y = ce^{kx}$. If one knows the initial condition, one can solve for c.

An equation of the form $y = e^x$ is a hyperbolic function. The slope of its curve changes very dramatically when plotted using Cartesian coordinates. That makes it very difficult, if not impossible, to solve the equation by plotting it on a standard graph. Fortunately, that problem can be overcome by plotting the data on semilogarithmic graph paper. A function of the type under discussion will yield a straight line when plotted on semilogarithmic paper. The semilogpaper is ruled using base 10 cycles (each cycle corresponds to 10 times the value of the previous cycle), but the differential equation is expressed in powers of e. The difference in expressions can be resolved by remembering that very important relation between log and ln: $\ln x = 2.3 \log x$. If that relation is not etched into your memory already, it is a good idea to have it written down in a convenient spot. Whenever you are plotting data on logarithmic or semilogarithmic graph paper in order to solve a differential equation expressed in terms of ln, you will need to substitute $2.3 \log x$ for each place that $\ln x$ appears. If you remember that

relation, it becomes a very simple matter to solve such problems graphically. Consider the differential equation $y\Delta t = \ln x_2 - \ln x_1$. This can be simplified to $y\Delta t = \ln x_2/x_1$, or $y = (2.3/\Delta t) \log x_2/x_1$. If $x_2 = 10x_1$, then the equation simplifies to $y = 2.3/\Delta t$, and you have only to read the difference in t across that same cycle in order to solve the problem.

There are also partial differential equations. LaPlace's equation, $\nabla^2 f(x, y, z) = 0$, is an example of a partial differential equation. It is of some importance in solving physics problems, so a few of its forms are given here:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \qquad (A2.5)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = 0 \qquad (A2.6)$$

$$\text{div grad } f = 0 \qquad (A2.7)$$

$$\nabla \cdot \nabla = 0 \qquad (A2.8)$$

The subject of partial differential equations is very involved, and well beyond the scope of this short mathematics review. The most important thing to remember is that partial differential equations are, as a rule, more complicated than ordinary differential equations, but they still have solutions.

A simple example follows. Find a solution for the LaPlace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \qquad (A2.9)$$

There are an infinite number of solutions for the equation as is stated in this simple form. A simple solution that should be intuitively evident is $x^2 + y^2 - 2z^2 = f(x, y, z)$. To verify, we find

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y, \qquad \frac{\partial f}{\partial z} = -4z \qquad (A2.10)$$

Then,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2, \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2, \qquad \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = -4 \qquad (A2.11)$$

so that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2 + 2 - 4 = 0 \qquad (A2.12)$$

establishing that our intuition was correct, and the equation is a solution to the LaPlace equation.

It should be evident that an infinite number of functions are solutions to the LaPlace equation. In fact, they form a subset of all possible functions. Keep in mind that LaPlace equations are not all as easily recognized as the one given above.

Vectors

When vectors are written out, they are conventionally given in terms of i, j, and k. These are the unit vectors in the three axis directions, and using i, j, and k avoids any confusion with the variables x, y, and z. If a vector can be expressed as $xi + yj + zk = v$, then $|v| = (x^2 + y^2 + z^2)^{1/2}$ is the magnitude of the vector. The dot product of two vectors u and v is $u \cdot v = |u||v| \cos \theta$, where θ is the angle between the two vectors. The dot product is therefore a scalar, or a nonvector number. The cross product of two vectors is another vector. If $u \times v = w$, then $|w| = |u||v|\sin \theta$, where θ is the angle between u and v. Suppose that $u = ai + bj + ck$ and $v = di + ej + fk$. Then, $u \cdot v = ad + be + cf$; $u \times v$ is more complicated.

It can be expressed as the minors of the determinant:

$$\begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} \qquad (A2.13)$$

which is $(bf - ec)i - (af - dc)j + (ae - db)k$.

Once again, this is intended as a very brief review of the subject. The review problems at the end of this appendix should help prepare you for the course, and help you remember prior mathematics course work. We will begin with a (fanciful) example of some theoretical mathematics, then some easy questions, followed by more difficult problems that come from actual experiments. If you run into difficulty, consult any mathematics book at your disposal.

THE TASTE FUNCTION

Suppose that you have decided to make blueberry pancakes for breakfast, and while doing so it occurs to you that there may be some "taste function" that could exactly describe how you want your pancakes prepared. Is there such a function? Think for a moment: If you could control all of the variables that you might think of, would your blueberry pancakes taste exactly the same each time that you made them? The answer should be yes, or at least that is the presumption made by cookbooks. If you used exactly the same ingredients, mixed together in exactly the same way in the same ratio, cooked in the same pan, over the same heat source for the same amount of time—if you controlled all of the variables—then you would certainly expect the same results each time. It is reasonable to make the beginning assumption that there is a taste function. We can test that assumption by designing a recipe that we will use each time. However, if the same recipe gives us two different results—if the pancakes taste wonderful one time and horrible the next—then it is not a function. (Unless, of course, you have a physiological problem that affected your taste buds, but that is just a problem with the subjective nature of measuring our taste function.)

Let us assume that there is such a taste function. We are all scientists; how would we describe the function in a mathematical sense? First, we would assign some representations for the variables that we believe control the taste of the blueberry pancakes. We could call the blueberries b, the pancake mix p, the milk m, and the eggs, e (this is a hypothetical mix, by the way). There will be a specific amount of each ingredient that will be used in the mix, and then the ingredients will be added together. So far this is simple algebra, but since we have not yet determined the amounts, the mix will be abbreviated as $f(b, m, p, e)$. Next, we will want to stir the mix. To simplify things a bit, assume that we are going to slowly pour the ingredients into a bowl while stirring, and that as we stir the mix, it is pouring out of a hole in the bottom of the bowl into a second bowl. The movement of the mix through the first bowl is a gradient field; you could theoretically measure its rate of movement with respect to any of the three axes. The rate of movement toward the second bowl at any one time at any one point would be the gradient, or $\nabla f(x, y, z)$. The mix would be rotated by our stirring it, so that there is a rotational component to the movement of the mix through the bowl. The amount of rotation about any point would be represented by $\nabla \times f(x, y, z)$. Would there be a $\nabla \cdot f$ component? Not really; the mix is constrained by the boundary of the bowl—it is prevented from spreading outward by the sides of the bowl. Suppose that we get to actually heating the pancake. The heat moves upward through the pancake, but it also moves outward as it's moving up. $\nabla \cdot f(x, y, z, T, t)$ is the theoretical expression for that vital component of the taste function.

You can see that we used a mathematical shorthand to describe what is actually a very simple act. If you were to string the components together to form a taste function, it would look very intimidating to a person unfamiliar to the procedure, or unfamiliar with the nature

of your variables—and this is not even a well-thought-out model! Do not let the strings of mathematical symbols intimidate you. Think of the equations as abstract shorthand rather than as an actual equation that you have to solve. These equations from vector calculus are just another scientist's way of expressing what he believes to be happening. Feel free to question the logic that was used by the scientist.

MATHEMATICS REVIEW PROBLEMS

Answers to these problems are given in the instructors solution manual

A2.1. How would you go about establishing that the following formula is, or is not, a function?

$$Q_c = \sum_{\alpha=1}^{\infty} -\frac{\overline{v_w}}{4\pi}\left[\int_0^{D^{(\alpha)}} \frac{D^{(\alpha)}}{\varepsilon}\, dD^{(\alpha)} - \frac{(D^{(\alpha)})^2}{2} - T\frac{\partial}{\partial T}\int_0^{D^{(v)}} \frac{D^{(\alpha)}}{\varepsilon}\, dD^{(\alpha)}\right]\frac{s \times 18}{(3.286\ \text{Å})^2\, N_A} \quad \text{(A2.14)}$$

where $D^{(\alpha)}$ is the electric displacement at phase α; ε is the dielectric constant of water; T is temperature; s is surface area per gram of sample (m^2g^{-1}); N_A is Avogadro's number (6.023×10^{23}); and v_w is the partial volume of water.

A2.2. You discover a scientific paper in which the author states that the ability of a soil to conduct water is dependent upon three factors: **(1)** The price of pretzels in Poughkeepsie; **(2)** the number of hairs on the bearded lady's chin; and **(3)** the number of goats on Old McDonald's Farm. Do you consider it likely that this really is a function? Why or why not? How would you go about proving or disproving it? (Although the topic may sound laughable, be as specific as possible in answering the last part of this question.)

A2.3. In order for an equation to be differentiable over an interval, it must be both a function, and continuous over the interval. (The difference between $\int$ and Σ is due to the difference between continuous and discrete phenomena. The amount of space contained within a jar would be determined by using $\int$, because one cannot "count" the space even though it is a very small number. The number of stars contained within the universe would be expressed by using Σ, because even though the number may be infinite, it is still countable—if you lived long enough and could count that high.) Most of the models that are used in this course are calculus-based, with plenty of dx, $\int$, ∂, ∇, and their friends. If you could slice a cross-section through the area of flow, would the flow of water through a boulder field be continuous? What about a layer of cobbles? Of gravel? Of sand? Obviously, there are an infinite number of points where the water flow could not be measured under each of these circumstances. Yet water flow is one of those factors that is described by using calculus. The question here is, are all of these scientists daft or is there some reason to use calculus to describe a discontinuous function? (This is not a mathematical question, but a logical one. Approach it as would a philosopher.)

A2.4. One of the most important measurable quantities encountered during this course is the hydraulic conductivity of soils—their ability to allow water to be transmitted through them. Henry Darcy designed an apparatus to measure K, the hydraulic conductivity, by filling an enclosed bed with sand and measuring how fast the water flowed through the pipe (chapter 7). What important mathematical operation was negated by this experimental design? Do you think that this might have some effect upon the experimental results?

A2.5. Darcy's experiment required that the soil be saturated in order to measure hydraulic conductivity. The measured result, called the saturated hydraulic conductivity, is represented by K_s. One method of measuring K_s is known as the "falling head method," (you will be spared the details of the method for now, but you can practice the mathematics that are involved). Suppose that $K_s = -a\,\Delta z\,dH/H\,dtA$, where a, Δz, and A are constants. Let H_0 be the value of H at t_0, and H_1 the value of H at t_1. Show that $K_s = 2.3a\,\Delta z/t_1 A$ when $H_0 = 10H_1$ and $t_0 = 0$. This is an ordinary differential equation. If you find this problem difficult you may want to review your mathematics a bit more. This type of differential equation is normally explained in the first or second semester of calculus.

A2.6. Let $\Delta z = 12$ cm, $a = 0.2$ cm^2, $A = 50$ cm^2, $H_0 = 30$ cm. You perform an experiment and obtain the following results.

t (in minutes)	1	5	10	15	20	30	40
H (cm)	28.8	27.3	25.0	23.1	21.1	17.6	15.0

Solve this by graphing the results and using the appropriate formula from problem A2.5. Express K_s in cm sec^{-1}.

A2.7. Another important property encountered in this course is diffusivity (see chapter 8). We will use the laboratory method of Bruce and Klute for obtaining diffusivity values to practice practical mathematics. The Bruce and Klute method uses the equation

$$D(\theta') = -\frac{1}{2t}\left(\frac{dx}{d\theta}\right)\int_{\theta_i}^{\theta_x} x\, d\theta \qquad (A2.15)$$

where t is time in seconds for the entire experiment to be performed; θ is the water content of the porous medium at a sample location (it is a percentage, and therefore unitless); and x is the distance (cm) of the sample from the water source. An experiment obtains the following results.

x	0	1	2	3	5	10	15	20	25	30	32	33	35
θ	.44	.43	.44	.43	.42	.40	.38	.36	.35	.30	.24	.18	.02

θ_i, the water content of the initial "dry" soil, is 0.002; t is forty-five minutes. What is the diffusivity ($D(\theta')$) when the soil has a water content of 40 percent? When it has a water content of 33 percent? 25 percent? (θ_x is shorthand for "the value of θ at x.") This problem is easy to solve if you remember the definitions of $\int$ and the derivative. Final hint: solve it graphically. Feel free to use the values of dy/dx, and a light table, if you need to.

A2.8. As an unsaturated-zone scientist with good mathematics skills, you have been asked to review the paper of a prominent scientist who made a slug test of a well in an unconfined aquifer. Being conscientious, you actually work through the mathematical formulas given, rather than simply peruse them and accept each at face value. Upon your initial reading, you detect that the author chose the wrong formula to integrate, of the two formulas given. You will need to simplify these formulas, and express both so that K is the dependent variable, and y is the independent variable. Underline in an introduction to your answer the basis for your selection of the correct formula that should be used, i.e., either A2.16 or A2.17.

The equilibrium equation for a confined aquifer is

$$T = \frac{Q}{2\pi[h_2 - h_1]}\ln\left(\frac{r_2}{r_1}\right) \qquad (A2.16)$$

and the equilibrium equation for an unconfined aquifer is

$$K = \frac{Q}{\pi[h_2^2 - h_1^2]}\ln\left(\frac{r_2}{r_1}\right) \qquad (A2.17)$$

where $T = Kb$; $h_2 - h_1 = y$ (and $h_2 = b$).

A2.9. **(a)** Integrate both equations given in problem A2.8; you will likely want to consult a table of integrals in order to do so. **(b)** What is the difference in the results given by the two equations (i.e., K will be understated by a substantial percentage—what is this percentage?) **(c)** When will the maximum difference in K between the two formulas occur?

Tables

COMPLEMENTARY ERROR FUNCTION (ERFC)

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp^{-\beta^2} d\beta$$

$$\mathrm{erf}(-z) = -\mathrm{erf}\, z$$

$$\mathrm{erfc}(z) = 1 - \mathrm{erf}(z)$$

z	$\mathrm{erf}(z)$	$\mathrm{erfc}(z)$	z	$\mathrm{erf}(z)$	$\mathrm{erfc}(z)$
0	0	1.0	1.1	0.880205	0.119795
0.05	0.056372	0.943628	1.2	0.910314	0.089686
0.1	0.112463	0.887537	1.3	0.934008	0.065992
0.15	0.167996	0.832004	1.4	0.952285	0.047715
0.2	0.222703	0.777297	1.5	0.966105	0.033895
0.25	0.276326	0.723674	1.6	0.976348	0.023652
0.3	0.328627	0.671373	1.7	0.983790	0.016210
0.35	0.379382	0.620618	1.8	0.989091	0.010909
0.4	0.428392	0.571608	1.9	0.992790	0.007210
0.45	0.475482	0.524518	2.0	0.995322	0.004678
0.5	0.520500	0.479500	2.1	0.997021	0.002979
0.55	0.563323	0.436677	2.2	0.998137	0.001863
0.6	0.603856	0.396144	2.3	0.998857	0.001143
0.65	0.642029	0.357971	2.4	0.999311	0.000689
0.7	0.677801	0.322199	2.5	0.999593	0.000407
0.75	0.711156	0.288844	2.6	0.999764	0.000236
0.8	0.742101	0.257899	2.7	0.999866	0.000134
0.85	0.770668	0.229332	2.8	0.999925	0.000075
0.9	0.796908	0.203092	2.9	0.999959	0.0000041
0.95	0.820891	0.179109	3.0	0.999978	0.000022
1.0	0.842701	0.157299			

SERIES EXPANSIONS

Exponential Integral

Let the argument of the exponential integral be x, as in $E_i(x)$. If x is less than or equal to 1.0, then

$$E_i(x) = -\ln x - 0.57721566 + 0.99999193*x - 0.24991055*x^2 + 0.5519968*x^3$$
$$- 0.00976004*x^4 + 0.00107857*x^5$$

If x is greater than 1.0, then

$$E_i(x) = \frac{[\exp{(-x)}/x]*[0.250621 + x*(2.334733 + x)]}{[1.681534 + x*(3.3306571 + x)]}$$

Error Function

Let the argument of the error function be x, as in $\mathrm{erf}(x)$. If x is greater than 3.0, then

$$\mathrm{erf}(x) = 1.0$$

If x is less than 0.0 (negative), then let $x = -x$; or if x is positive but less than or equal to 0.9, then let $x = x$. Also let $y = x^2$ for x negative or positive. Then,

$$\mathrm{erf}(x) = 1.12838*x*((((((-7.57576E - 4*y + 0.00462963)*y$$
$$ - 0.0238095)*y + 0.01)*y) - 0.333333)*y + 1.0)$$

and, if x was negative, then

$$\mathrm{erf}(x) = -\mathrm{erf}(x)$$

If x is greater than 0.9 but less than or equal to 3.0, then $y = x^2$ and

$$T = \frac{1.0}{1.0 + 0.47047*x}$$

Then,

$$\mathrm{erf}(x) = 1.0 - T*((0.7478556*T - 0.0958798)*T + 0.3480242)*\exp(-y)$$

CONVERSION FACTORS

The student may find these conversion units useful for problems encountered in the text, and elsewhere in unsaturated zone hydrology. Most of the values in the following conversion tables have been rounded to the second decimal. For greater accuracy, consult a recent issue of the *CRC Handbook of Chemistry and Physics*. To convert a value from the units in the "From" column to those in the "To" column, multiply the "From" column by the numerical value at the intersection.

Example: 78 gal $\min^{-1}$ to $m^3\ hr^{-1}$: 78 gal $\min^{-1}*0.227 = 17.707\ m^3\ hr^{-1}$; or, in reverse: $17.707\ m^3\ hr^{-1}*4.404 = 78$ gal $\min^{-1}$.

			Flow Rate Conversion			
From → To	L sec^{-1}	m^3 min^{-1}	**m^3 hr^{-1}**	gal min^{-1}	gal hr^{-1}	ft^3 sec^{-1}
L sec^{-1}	1	0.06	3.60	15.85	9.51×10^2	3.53×10^{-2}
m^3 min^{-1}	16.67	1	60	2.64×10^2	1.59×10^4	0.589
m^3 hr^{-1}	0.28	1.67×10^{-2}	1	4.404	2.64×10^2	9.81×10^{-3}
gal min^{-1}	6.31×10^{-2}	3.79×10^{-3}	**0.227**	1	60	2.23×10^{-3}
gal hr^{-1}	1.05×10^{-3}	6.31×10^{-5}	3.79×10^{-3}	1.67×10^{-2}	1	3.71×10^{-5}
ft^3 sec^{-1}	28.32	1.70	1.02×10^2	4.49×10^2	2.69×10^4	1

Concentration Conversion

From → To	mg L^{-1}	g L^{-1}	kg m^{-3}	lb in.$^{-3}$	lb ft^{-3}	lb gal^{-1}	grain gal^{-1}
mg L^{-1}	1	1×10^{-3}	1×10^{-3}	3.61×10^{-8}	6.24×10^{-5}	8.35×10^{-6}	5.84×10^{-2}
g L^{-1}	1×10^{3}	1	1	3.61×10^{-5}	6.24×10^{-2}	8.35×10^{-3}	58.42
kg m^{-3}	1×10^{3}	1	1	3.61×10^{-5}	6.24×10^{-2}	8.35×10^{-3}	58.42
lb in.$^{-3}$	2.77×10^{7}	2.77×10^{4}	2.77×10^{4}	1	1.73×10^{3}	2.31×10^{2}	1.62×10^{6}
lb ft^{-3}	1.60×10^{4}	16.02	16.02	5.79×10^{-4}	1	0.134	9.35×10^{2}
lb gal^{-1}	1.20×10^{5}	1.20×10^{2}	1.20×10^{2}	4.33×10^{-3}	7.48	1	7.0×10^{3}
grain gal^{-1}	17.12	1.71×10^{-2}	1.71×10^{-2}	6.19×10^{-7}	1.07×10^{-3}	1.43×10^{-4}	1

Length Conversion

From → To	micron	mm	cm	m	km	in.	ft	mile
micron	1	1×10^{-3}	1×10^{-4}	1×10^{-6}	1×10^{-9}	3.94×10^{-5}	3.28×10^{-6}	6.21×10^{-10}
mm	1×10^{3}	1	1×10^{-1}	1×10^{-3}	1×10^{-6}	3.94×10^{-2}	3.28×10^{-3}	6.21×10^{-7}
cm	1×10^{4}	10	1	1×10^{-2}	1×10^{-5}	0.394	3.28×10^{-2}	6.21×10^{-6}
m	1×10^{6}	1×10^{3}	1×10^{2}	1	1×10^{-3}	39.37	3.28	6.21×10^{-4}
km	1×10^{9}	1×10^{6}	1×10^{5}	1×10^{3}	1	3.94×10^{4}	3.28×10^{3}	0.621
in.	2.54×10^{4}	25.40	2.54	2.54×10^{-2}	2.54×10^{-5}	1	8.33×10^{-2}	1.58×10^{-5}
ft	3.05×10^{5}	3.05×10^{2}	30.48	0.305	3.05×10^{-4}	12	1	1.89×10^{-4}
mile	1.61×10^{9}	1.61×10^{6}	1.61×10^{5}	1.61×10^{3}	1.61	6.34×10^{4}	5280	1

Area Conversion

From → To	mm^2	cm^2	m^2	km^2	ha	in.^{2}	ft^2	acre	mile^2
mm^2	1	1×10^{-2}	1×10^{-6}	1×10^{-12}	1×10^{-10}	1.55×10^{-3}	1.08×10^{-5}	2.47×10^{-10}	3.86×10^{-13}
cm^2	1×10^{2}	1	1×10^{-4}	1×10^{-10}	1×10^{-8}	0.155	1.08×10^{-3}	2.47×10^{-8}	3.86×10^{-11}
m^2	1×10^{6}	1×10^{4}	1	1×10^{-6}	1×10^{-4}	1.55×10^{3}	10.76	2.47×10^{-4}	3.86×10^{-7}
km^2	1×10^{12}	1×10^{10}	1×10^{6}	1	1×10^{2}	1.55×10^{9}	1.08×10^{7}	2.47×10^{2}	0.386
ha	1×10^{10}	1×10^{8}	1×10^{4}	1×10^{-2}	1	1.55×10^{7}	1.08×10^{5}	2.47	3.86×10^{-3}
in.^{2}	6.45×10^{2}	6.45	6.45×10^{-4}	6.45×10^{-10}	6.45×10^{-8}	1	6.94×10^{-3}	1.59×10^{-7}	2.49×10^{-10}
ft^2	9.29×10^{4}	9.29×10^{2}	9.29×10^{-2}	9.29×10^{-8}	9.29×10^{-6}	144	1	2.30×10^{-5}	3.59×10^{-8}
acre	4.05×10^{9}	4.05×10^{7}	4.05×10^{3}	4.05×10^{-3}	0.405	6.27×10^{6}	4.36×10^{4}	1	1.56×10^{-3}
mile^2	2.59×10^{12}	2.59×10^{10}	2.59×10^{6}	2.59	2.59×10^{2}	4.01×10^{9}	2.79×10^{7}	640	1

Volume Conversion

From → To	cm³	L	m³	in.³	ft³	pint	qt	gal	ac-ft
cm3	1	1×10^{-3}	1×10^{-6}	6.10×10^{-2}	3.53×10^{-5}	2.11×10^{-3}	1.06×10^{-3}	2.64×10^{-4}	8.11×10^{-10}
L	1×10^{3}	1	1×10^{-3}	61.02	3.53×10^{-2}	2.11	1.06	0.264	8.11×10^{-7}
m³	1×10^{6}	1×10^{3}	1	6.10×10^{4}	35.31	2.11×10^{3}	1.06×10^{2}	2.64×10^{2}	8.11×10^{-4}
in.³	16.39	1.64×10^{-2}	1.64×10^{-5}	1	5.79×10^{-4}	3.46×10^{-2}	1.73×10^{-2}	4.33×10^{-3}	1.33×10^{-8}
ft³	2.83×10^{4}	28.32	2.83×10^{-2}	1.73×10^{3}	1	59.84	29.92	7.48	2.30×10^{-5}
pint	4.73×10^{2}	0.473	4.73×10^{-4}	28.87	1.67×10^{-2}	1	0.500	0.125	3.84×10^{-7}
qt	9.46×10^{2}	0.946	9.46×10^{-4}	57.74	3.34×10^{-2}	2	1	0.250	7.67×10^{-7}
gal	3.79×10^{3}	3.785	3.79×10^{-3}	2.31×10^{2}	0.134	8	4	1	3.07×10^{-6}
ac-ft	1.23×10^{9}	1.23×10^{6}	1.23×10^{3}	7.52×10^{7}	4.35×10^{4}	2.61×10^{6}	1.30×10^{6}	3.26×10^{5}	1

Mass Conversion

From → To	mg	g	kg	metric ton	oz.	lb	short ton	grain
mg	1	1×10^{-3}	1×10^{-6}	1×10^{-9}	3.53×10^{-5}	2.21×10^{-6}	1.10×10^{-9}	1.54×10^{-2}
g	1×10^{3}	1	1×10^{-3}	1×10^{-6}	3.53×10^{-2}	2.21×10^{-3}	1.10×10^{-6}	15.43
kg	1×10^{6}	1×10^{3}	1	1×10^{-3}	35.27	2.21	1.10×10^{-3}	1.54×10^{4}
metric ton	1×10^{9}	1×10^{6}	1×10^{3}	1	3.53×10^{4}	2.21×10^{3}	1.10	1.54×10^{7}
oz.	2.84×10^{4}	28.35	2.84×10^{-2}	2.84×10^{-5}	1	6.25×10^{-2}	3.13×10^{-5}	437.5
lb	4.54×10^{5}	453.6	0.454	4.54×10^{-4}	16	1	5.00×10^{-4}	7.00×10^{3}
short ton	9.07×10^{5}	9.07×10^{5}	9.07×10^{2}	0.907	3.20×10^{4}	2000	1	1.40×10^{7}
grain	64.80	6.48×10^{-2}	6.48×10^{-5}	6.48×10^{-8}	2.29×10^{-3}	1.43×10^{-4}	7.14×10^{-8}	1

Pressure Conversion

From → To	dyne cm⁻²	kg m⁻²	Pa	cm Hg	ft H₂O	bar	atm	psi	lb ft⁻²
dyne cm⁻²	1	1.01×10^{-2}	0.100	7.50×10^{-5}	3.35×10^{-5}	1×10^{-6}	9.87×10^{-7}	1.45×10^{-5}	2.09×10^{-3}
kg m⁻²	98.07	1	9.81	7.36×10^{-3}	3.28×10^{-3}	9.81×10^{-5}	9.68×10^{-5}	1.42×10^{-3}	0.205
Pa	10.00	0.102	1	7.50×10^{-4}	3.35×10^{-4}	1×10^{-5}	9.87×10^{-6}	1.45×10^{-5}	2.09×10^{-2}
cm Hg	1.33×10^{4}	1.36×10^{2}	1.33×10^{3}	1	0.446	1.33×10^{-2}	1.32×10^{-2}	0.193	27.85
ft H₂O	2.99×10^{4}	3.05×10^{2}	2.99×10^{3}	2.241	1	2.99×10^{-2}	2.95×10^{-2}	0.434	62.42
bar	1×10^{6}	1.02×10^{4}	1×10^{5}	75.01	33.46	1	0.987	14.50	2.09×10^{3}
atm	1.01×10^{6}	1.03×10^{4}	1.01×10^{5}	76.00	33.90	1.013	1	14.70	2.12×10^{3}
psi	6.90×10^{4}	7.03×10^{2}	6.90×10^{3}	5.171	2.307	6.90×10^{-2}	6.80×10^{-2}	1	144
lb ft⁻²	4.79×10^{2}	4.882	47.88	3.59×10^{-2}	1.60×10^{-2}	4.79×10^{-4}	4.73×10^{-4}	6.94×10^{-3}	1

Physical Constants

Atomic mass unit	u	1.661×10^{-27} kg
Avogadro constant	N_A	6.022×10^{23} mol^{-1}
Boltzmann constant	k	1.381×10^{-23} J K^{-1}
Electron rest mass	m_e	9.100×10^{-31} kg
Elementary charge	e	1.602×10^{-19} C (electric charge per mol of electrons)
Electron charge	$-e/m_e$	-1.7588×10^{11} C kg^{-1} or 4.803×10^{-10} abs esu
Faraday constant	F	9.6485×10^4 C mol^{-1} $23{,}060$ cal mol^{-1} eV^{-1}
Gas constant	R	8.314 J K^{-1} mol^{-1} 0.08314 L bar K^{-1} mol^{-1} 1.987 cal K^{-1} mol^{-1} 0.08206 L atm K^{-1} mol^{-1}
Ice point (absolute zero)		273.15 K
Molar volume (ideal gas, 0 °C, 1 atm)	V_m	22.414×10^3 cm^3 mol^{-1}
Natural logarithm of 10	$\ln 10$	2.302585
Permittivity of vacuum	ε_o	8.854×10^{-12} C^2 N^{-1} m^{-2}
	ε_o	8.854×10^{-12} C^2 J^{-1} m^{-2}
	$1/4\pi\varepsilon_o$	0.8988×10^{10} N m^2 C^{-2}
Planck constant	h	6.626×10^{-34} J s
Proton rest mass	m_p	1.673×10^{-27} kg
$R \ln 10$		19.14 J mol^{-1} K^{-1}
$RT_{298.15} \ln \chi$		$5706.6 \log \chi$ J mol^{-1} or $1364.1 \log \chi$ cal mol^{-1}
$RTF^{-1} \ln 10$		59.16 mV at 298.15 K
$RTF^{-1} \ln \chi$		$0.05916 \log \chi$, volt at 298.15 K
Rydberg constant	R_∞	1.097×10^7 m^{-1}
Speed of light in vacuum	c	2.998×10^8 m s^{-1}

Useful Conversion Factors

Energy, work, heat 1 joule = 1 volt-coulomb = 1 newton meter
= 1 watt-second = 2.7778×10^{-7} kilowatt hours
= 10^7 erg
= 9.9×10^{-3} liter atmospheres
= 0.239 calorie
= 1.0365×10^{-5} volt-faraday
= 6.242×10^{18} eV
= 5.035×10^{22} cm^{-1} (wave number)
= 9.484×10^{-4} BTU (British thermal unit)
= 3×10^{-8} kg coal equivalent

Entropy   1 entropy unit, $\text{cal mol}^{-1}\,\text{K}^{-1} = 4.184\ \text{J mol}^{-1}\,\text{K}^{-1}$

Power $1\ \text{watt} = 1\ \text{kg m}^2\,\text{s}^{-3}$

$\qquad\qquad\qquad = 2.39 \times 10^{-4}\ \text{kcal s}^{-1} = 0.860\ \text{kcal h}^{-1}$

Pressure $1\ \text{atm} = 760\ \text{torr} = 760\ \text{mm Hg}$

$\qquad\qquad\qquad = 1.013 \times 10^5\ \text{N m}^{-2} = 1.013 \times 10^5\ \text{Pa (Pascal)}$

$\qquad\qquad\qquad = 1.013\ \text{bar}$

Statistics on water

Density @ 3.98 °C (peak density) = $999.973\ \text{kg m}^{-3}$ = $0.999973\ \text{g cm}^{-3}$ = $1.000000\ \text{g ml}^{-1}$

Density @ 0 °C = $999.87\ \text{kg m}^{-3}$

Ice Density @ 0 °C = $916.76\ \text{kg m}^{-3}$

Converting X number of ppm$_\text{V}$ to density (g m^{-3}) units

At STP, 1 mole of gas occupies $22.4\ \text{l}\ (V_o)$, at $P_o = 101.325\ \text{kPa}$, $T_o = 273.16\ \text{K}$, where a mole of the particular gas has a mass (m_c) in grams/mole.

From the ideal gas law:

$$P_o V_o = m_c R_c T_o \qquad \text{and} \qquad \rho = \rho_c R_c T$$

Using both equations, solving for ρ_c in terms of volume, and inserting constants:

$$\rho_c[\text{gm}^{-3}] = \left(\frac{X\ \text{parts volume}}{10^6\ \text{parts volume}}\right)\left(\frac{m_c[\text{gm mol}^{-1}]}{22.4\ [\text{L mol}^{-1}]}\right)\left(\frac{1000[\text{L}]}{[\text{m}^3]}\right)\left(\frac{273.15\ \text{K}}{T\ [\text{K}]}\right)\left(\frac{P[\text{kPa}]}{101.325\ \text{kPa}}\right)$$

where "L" refers to liters.

Converting water in chemical potential units [J kg^{-1}] to pressure potential units [Pa]

Often it is more convenient to work with water potential in pressure units, particularly if water potential is sensed with pressure transducers. Here, a chemical potential of A in [J kg^{-1} or energy per unit mass] is converted to pressure potential [Pa] units by dividing the energy per mass units by the partial molal volume of water ($18 \times 10^{-6}\ \text{m}^3\ \text{mol}^{-1}$), valid for dilute solutions.

$$\frac{A[\text{J kg}^{-1}]}{18 \times 10^{-6}[\text{m}^3\ \text{mol}^{-1}]} \times 0.018016\ [\text{kg mol}^{-1}] = A[\text{kPa}]$$

Converting X number of pCi (pico Curies) of a radioactive gas to its concentration in parts per million (ppm$_\text{V}$) units

Radioactive materials decay at a rate given by the following simple, ordinary differential equation:

$$\frac{dS}{dt} = -kS$$

where S is concentration, t is time, and k is the disintegration constant. Radioactive half-life is defined as the amount of time for half of the mass of radioactive material to decay. Solving the above equation, we can solve for the disintegration constant using half-life:

$$\frac{S}{S_o} = \frac{1}{2} = \exp(-kt)$$

where S_o is the initial concentration of element S. The disintegration rate per unit time $[d\,s^{-1}]$ is measured in Curies (C): $C = 3.7 \times 10^{10}\,d\,s^{-1}$ or in pico Curies: $pC = 0.037\,d\,s^{-1}$. The conversion from $pC\,L^{-1}$ to ppm follows from rearrangement of the above equations:

$$\frac{\frac{dS}{dt}}{k} = S = \left(\frac{\frac{X[pC]}{[L]}}{k[s^{-1}]}\right)\left(\frac{0.037[d\,s^{-1}]}{pC}\right)$$

which yields the result of disintegrations per liter. Since there is one disintegration per atom, the final conversion to ppm_v is:

$$\left(\frac{S\,[\text{atoms}]}{[L]}\right)\left(\frac{1\,[\text{mole}]}{6.023 \times 10^{23}\,[\text{atoms}]}\right)\left(\frac{22.4\,[L]}{[\text{mole}]}\right)\left(\frac{10^6\,\text{parts}\,[L]}{1\,\text{part}\,[L]}\right) = y\,ppm_v$$

This conversion is at STP (standard temperature and pressure).

A conversion example: Radon has a half-life of 3.82 days $= 3.3 \times 10^5$ s. With t $= 3.3 \times 10^5$ s, k $= 2.10 \times 10^{-6}$ s^{-1}. If radon concentration is 100 pC l^{-1}, one may calculate a 1.74×10^6 disintegrations per liter and a concentration of 6.5×10^{-11} ppm$_v$.

THE INTERNATIONAL SYSTEM OF UNITS (SI)

This unit system is based on the metric system. It was designed to achieve maximum internal consistency and based upon the following defined units:

Physical quantity	Unit	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Luminous intensity	candela	cd
Amount of material	mole	mol

The main derived units are:

Force	newton	$N\,(kg\,m^{-2})$
Energy, work, heat	joule	$J\,(N\,m)$
Pressure	pascal	$Pa\,(N\,m^{-2})$
Power	watt	$W\,(J\,s^{-1})$
Electric charge	coulomb	$C\,(As)$
Electric potential	volt	$V\,(W\,A^{-1})$
Electric capacitance	farad	$F\,(As\,V^{-1})$
Electric resistance	ohm	$\Omega\,(V\,A^{-1})$
Frequency	hertz	$Hz\,(s^{-1})$
Conductance	siemens	$S\,(A\,V^{-1})$

To convert SI units into Non-SI units multiply by	SI unit	Non-SI units	To convert Non-SI units into SI units, multiply by
		Energy, work, quantity of heat	
9.52×10^{-4}	joule, J	British thermal unit, BTU	1.05×10^{3}
0.239	joule, J	calorie, cal	4.19
10^{7}	joule, J	erg	10^{-7}
0.735	joune, J	foot-pound	1.36
2.387×10^{-5}	joule per square meter, J m^{-2}	calorie per square centimeter (langley)	4.19×10^{4}
10^{5}	newton, N	dyne	10^{-5}
1.43×10^{-3}	watt per square meter, W m^{-2}	calorie per square centimeter minute irradiance, cal cm^{-2} min^{-1}	698
		Transpiration and photosynthesis	
3.60×10^{-2}	milligram per square meter second, mg m^{-2} s^{-1}	gram per square decimeter hour, g dm^{-2} h^{-1}	27.8
5.56×10^{-3}	milligram(H$_2$O) per square meter second, mg m^{-2} s^{-1}	micromole (H$_2$O) per square centimeter second, μmol cm^{-2} s^{-1}	180
10^{-4}	milligram per square meter second, mg m^{-2} s^{-1}	milligram per square centimeter second, mg m^{-2} s^{-1}	10^{4}
35.97	milligram per square meter second, mg m^{-2} s^{-1}	milligram per square decimeter hour, mg dm^{-2} h^{-1}	2.78×10^{-2}
		Water measurement and flow rate	
9.73×10^{-3}	cubic meter, m^{3}	acre-inches, acre-in.	102.8
9.81×10^{-3}	cubic meter per hour, m^{3} h^{-1}	cubic feet per second, ft^{3} s^{-1}	101.9
4.40	cubic meter per hour, m^{3} h^{-1}	U.S. gallons per minute, gal min^{-1}	0.227
2.642×10^{2}	cubic meter per hour, m^{3} h^{-1}	U.S. gallons per hour, gal hr^{-1}	3.785×10^{-3}
8.11	hectare-meters, ha-m	acre-feet, acre-ft	0.123
97.28	hectare-meters, ha-m	acre-inches, acre-in.	1.03×10^{-2}
8.1×10^{-2}	hectare-centimeters, ha-cm	acre-feet, acre-ft	12.33
		Concentrations	
1	centimole per kilogram, cmol kg^{-1} (ion exchange capacity)	milliequivalents per 100 grams, meq 100 g^{-1}	1
0.1	gram per kilogram, g kg^{-1}	percent, %	10
1	milligram per kilogram, mg kg^{-1}	parts per million, ppm	1
		Electrical conductivity, electricity, magnetism	
10	siemen per meter, S m^{-1}	millimho per centimeter, mmho cm^{-1}	0.1
10^{4}	tesla, T	gauss, G	10^{-4}
		Radioactivity	
2.7×10^{-11}	becquerel, Bq	curie, Ci	3.7×10^{10}
2.7×10^{-2}	becquerel per kilogram, Bq kg^{-1}	picocurie per gram, pCi g^{-1}	37
100	gray, Gy (absorbed dose)	rad, rd	0.01
100	sievert, Sv (equivalent dose)	rem (roentgen equivalent man)	0.01
		Plane Angle	
57.3	radian, rad	degrees (angle), °	1.75×10^{-2}

To convert SI units into Non-SI units multiply by	SI unit	Non-SI units	To convert Non-SI units into SI units, multiply by
Length			
0.621	kilometer, km (10^3 m)	mile, mi	1.609
1.094	meter, m	yard, yd	0.914
3.28	meter, m	foot, ft	0.304
1.0	micrometer, μm (10^{-6} m)	micron, μ	1.0
3.94×10^{-2}	millimeter, mm (10^{-3} m)	inch, in.	25.4
10	nanometer, nm (10^{-9} m)	angstrom, Å	0.1
Area			
2.47	hectare, ha	acre	0.405
247	square kilometer, km^2 (10^3 m)^2	acre	4.05×10^{-3}
0.386	square kilometer, km^2 (10^3 m)^2	square mile, mi^2	2.590
2.47×10^{-4}	square meter, m^2	acre	4.05×10^3
10.76	square meter, m^2	square foot, ft^2	9.29×10^{-2}
1.55×10^{-3}	square millimeter, mm^2 (10^{-3} m)^2	square inch, in.^{2}	645
Volume			
9.73×10^{-3}	cubic meter, m^3	acre-inch, acre in	102.8
35.3	cubic meter, m^3	cubic foot, ft^3	2.83×10^{-2}
6.10×10^4	cubic meter, m^3	cubic inch, in.^{3}	1.64×10^{-5}
2.84×10^{-2}	liter, L (10^{-3} m^3)	bushel, bu	35.24
1.057	liter, L (10^{-3} m^3)	quart (liquid), qt	0.946
3.53×10^{-2}	liter, L (10^{-3} m^3)	cubic foot, ft^3	28.3
0.265	liter, L (10^{-3} m^3)	gallon, gal	3.78
33.78	liter, L (10^{-3} m^3)	ounce (fluid), oz	2.96×10^{-2}
2.11	liter, L (10^{-3} m^3)	pint (fluid), pt	0.473
Mass			
2.20×10^{-3}	gram, g (10^{-3} kg)	pound, lb	454
3.52×10^{-2}	gram, g (10^{-3} kg)	ounce (avdp), oz	28.4
2.205	kilogram, kg	pound, lb	0.454
0.01	kilogram, kg	quintal (metric), q	100
1.10×10^{-3}	kilogram, kg	ton (2000 lb), ton	907
1.102	megagram, Mg (tonne)	ton (U.S.), ton	0.907
1.102	tonne, t	ton (U.S.), ton	0.907
Yield and Rate			
0.893	kilogram per hectare, kg ha^{-1}	pound per acre, lb acre^{-1}	1.12
7.77×10^{-2}	kilogram per cubic meter, kg m^{-3}	pound per bushel, lb bu^{-1}	12.87
1.49×10^{-2}	kilogram per hectare, kg ha^{-1}	bushel per acre, 60 lb	67.19
1.59×10^{-2}	kilogram per hectare, kg ha^{-1}	bushel per acre, 56 lb	62.71
1.86×10^{-2}	kilogram per hectare, kg ha^{-1}	bushel per acre, 48 lb	53.75
0.107	liter per hectare, L ha^{-1}	gallon per acre, gal acre^{-1}	9.35
893	tonnes per hectare, t ha^{-1}	pound per acre, lb acre^{-1}	1.12×10^{-3}
893	megagram per hectare, Mg ha^{-1}	pound per acre, lb acre^{-1}	1.12×10^{-3}
0.446	megagram per hectare, Mg ha^{-1}	ton (2000 lb) per acre, ton acre^{-1}	2.24
2.24	meter per second, m s^{-1}	mile per hour, mph	0.447
Specific Surface Area			
10	square meter per kilogram, m^2 kg^{-1}	square centimeter per ram, cm^2 g^{-1}	0.1
1000	square meter per kilogram, m^2 kg^{-1}	square millimeter per gram, mm^2 g^{-1}	0.001

To convert SI units into Non-SI units multiply by	SI unit	Non-SI units	To convert Non-SI units into SI units, multiply by
		Pressure	
9.90	megapascal, MPa (10^6 Pa)	atmosphere	0.101
10	megapascal, MPa (10^6 Pa)	bar	0.1
1.00	megagram, Mg m^{-3}	gram per cubic centimeter, g cm^{-3}	1.00
2.09×10^{-2}	pascal, Pa	pound per square foot, lb ft^{-2}	47.9
1.45×10^{-4}	pascal, Pa	pound per square inch, lb in.$^{-2}$	6.90×10^3
		Temperature	
1.00 (K-273)	kelvin, K	Celsius, °C	1.00 (°C + 273)
(9/5 °C) + 32	Celsius, °C	Fahrenheit, °F	5/9(°F−32)

SI Units for Use in Unsaturated Zone Hydrology

Quantity	Application	Symbol
Concentration	Gas concentration	g m^{-3}
		mol m^{-3}
	Water content	kg kg^{-1}
		m^3 m^{-3}
Density	Particle density	Mg m^{-3}
	Bulk density	
Flux density	Heat flow	W m^{-2}
	Gas diffusion	g m^{-2} s^{-1}
		mol m^{-2} s^{-1}
	Water flow	kg m^{-2} s^{-1}
		m^3 m^{-2} s^{-1}
Gas diffusivity	Gas diffusion	m^2 s^{-2}
Hydraulic conductivity	Water flow	kg s m^{-3}
		m^3 s^{-1}
		m s^{-1}
Potential energy of soil water	Driving force for flow	J kg^{-1}
		kPa
		m
Specific heat	Heat storage	J kg^{-1} K^{-1}

Units for Water Flow Applied to Darcy's Law

Flux density	Hydraulic conductivity	Potential gradient
Potential in energy per unit mass (kg m^{-2} s^{-1})	kg s m^{-3}	J kg^{-1} m^{-1}
Potential in energy per unit volume (m s^{-1})	m^3 s kg^{-1}	Pa m^{-1}
Potential in energy per unit weight (m s^{-1})	m s^{-1}	m m^{-1}

Source: Data from Campbell and Schilfgaarde (1981)

LIST OF SYMBOLS

Symbol	Description	Units
a	Constant or specific volume or albedo	L^3 or decimal
a	Activity	—
a	Sphere radius	m
A	Available energy	$W\,m^{-2}$
A	Measured Carbon-14 radioactivity	—
A	Area	m^2
A	Specific surface area	$m^2\,kg^{-1}$
A	Hamaker constant (see equation 3.48)	ergs
a_i	Activity of species i	$mol\,L^{-1}$
A_o	Wave amplitude	—
A_o	Radioactivity level at some initial time	Bq
A_s	Interstitial surface area of pores	$m^2\,kg^{-1}$
b	Constant	—
b	Complex number	—
b	Width of sample	m
B	Retardation of the soil being used	$J\,m^{-2}$
c	Specific heat of soil	$J\,kg^{-1}\,K^{-1}$
c	Solute concentration	$mol\,L^{-1}$
^{12}C	Carbon 12	—
^{13}C	Carbon 13	—
^{14}C	Carbon 14	—
c	Integration constant	—
C	Curies or concentration	$(3.7 \times 10^{10}\,Bg)$ or $mg\,L^{-1}$
C	Ion concentration at specified distance from charged surface	$ions\,cm^{-3}$
C_e	Equilibrium aqueous concentration	$mg\,L^{-1}$
C_{gw}	Contaminant concentration in ground water	$\mu g\,L^{-1}$
C_p	Concentration of pollutant	$\mu g\,L^{-1}$
C_i	Specific heat of soil constituent i	$J\,kg^{-1}\,K^{-1}$
C_i	Concentration of species i	$mol\,L^{-1}$
C_o	Ion concentration in bulk solution	$ions\,cm^{-3}$
C_p	Specific heat of air	$J\,kg^{-1}\,K^{-1}$
C_p^o	Heat capacity	$J\,mol^{-1}$
C_s	Molar salt concentration	$M\,L^{-3}$
C_{soil}	Specific heat of soil	$J\,kg^{-1}\,K^{-1}$
d	Depth of soil layer	m
d	Distance, diameter, thickness, or midpoint	m
D	Fractal dimension	—
D	Mass of water lost through drainage out of soil column	kg
D	Hydraulic diffusivity	$L^2\,T^{-1}$
D_H	Hydrodynamic dispersion coefficient	$L^2\,T^{-1}$
D	Width of crack; $2d$ of spherical particle	m
D	Dielectric constant	unitless
D_a	Diffusivity of air	$m\,s^{-1}$
D_{ij}	Binary gas diffusion coefficient	$m\,s^{-1}$
D_i^k	Knudsen diffusion coefficient	$m\,s^{-1}$
D_h	Thermal diffusivity of soil	$m\,s^{-1}$
D_S	Diffusion coefficient of gas species s	$m\,s^{-1}$
D_T	Thermal diffusivity for thermally driven moisture flux	$m\,s^{-1}$
D_V	Molecular diffusivity of water vapor	$m\,s^{-1}$
D_α	Diffusion coefficient in free air	$m\,s^{-1}$
$D^{(\alpha)}$	Electric displacement at phase α	$C\,m^{-2}$
D_θ	Diffusivity for moisture flux due to a moisture gradient	$m\,s^{-1}$
d_j	Thickness of layer	m
d_t	Total depth of considered soil	cm

(continued)

Symbol	Description	Units
d_w	Equivalent depth of soil water if extracted and ponded over soil surface	cm
e	Unit of electronic charge (cgs or esu system)	$4.803*10^{-10}$ esu
e	Unit of electronic charge (SI system)	$1.6021*10^{-19}$ C
e	Void ratio	unitless
e	Vapor pressure	Pa
erf	Error function	—
erfc	Complimentary error function	—
e_s	Saturation vapor pressure	Pa
e_{sw}	Saturated vapor pressure at wet bulb temp.	Pa
E	Evaporation rate	$m\,s^{-1}$
E	Water vapor flux (mass flux or evaporation rate)	$g\,m^{-2}\,s^{-1}$
E	Potential energy per unit volume	$J\,m^{-3}$
E	Young's modulus	$N\,m^{-2}$
E	Electric field strength	N or $Kg\,m\,s^{-2}$
E_c	Charge-balance error (deviation from electroneutrality)	unitless
E_{cum}	Cumulative rate evaporation	$m\,s^{-1}$
E_h	Redox potential	V
E_i	Potential energy of interaction	$erg\,cm^{-2}$
E_{max}	Maximum rate of evaporation	$m\,s^{-1}$
E_o	Free-water evaporation rate	$m\,s^{-1}$
ET	Evapotranspiration (can also use flux units $g\,m^{-2}\,s^{-1}$)	$m\,s^{-1}$
ET_p	Potential evapotranspiration	$m\,s^{-1}$
ET_{eq}	Equilibrium evapotranspiration	$m\,s^{-1}$
e^o	Saturated water vapor pressure	Pa
E^o	Standard redox potential	V
E_s	Induced streaming potential by a thermal gradient	$V\,m^{-1}$
E_s	Electric resistivity	$S\,m^{-1}$
exp	Exponential function	—
f	Fractional amount	decimal
f	Frictional coefficient	unitless
f	Area of the air–water interface associated with the triangular volume	m^2
f	Force or frictional resistance	dynes or N or $Kg\,m\,s^{-2}$
f_i	Volume fraction of a soil constituent	—
F	Specific flux	$g\,m^{-2}\,s^{-1}$
F	Faraday constant	$96490\ C\,mol^{-1}$
F	Partial molal free energy	$J\,g^{-1}$
F	Free energy	$J\,mol^{-1}$ or $ergs\,mol^{-1}$
F	Linear free energy	$J\,mol^{-1}$
F_b	Breaking force applied at the center of the soil sample beam span	N or $Kg\,m\,s^{-2}$
F_c	Critical flocculation concentration value	$mol\,L^{-1}$
F_d	Resistence or drag force on an individual particle	$g\,cm^2\,sec^{-2}$ (dynes)
F^d	Drag force	N or $Kg\,m\,s^{-2}$
F_g	Force of gravity on a soil particle	$g\,cm^2\,sec^{-2}$
F_i	Molar flux of constituent i	$mol\,m^{-2}\,s^{-1}$
F_i^s	Static equilibrium	—
F^s	Static force	N or $Kg\,m\,s^{-2}$
F_w	Partial molal free energy under influence of forces	$J\,g^{-1}$
g	Gravitational acceleration	$m\,s^{-2}$
g	Gravitational constant	$N\,m^2\,kg^{-2}$
G	Soil heat flux	$W\,m^{-2}$
G	Strain energy release rate	$dynes\,cm^{-2}$ or $J\,m^{-2}$
G_C	Change in stored heat	$W\,m^{-2}$
G^E	Excess Gibbs free energy	$J\,mol^{-1}$
G^I	Gibbs free energy of mixing	$J\,mol^{-1}$

(*continued*)

Symbol	Description	Units
G^M	Energy of mixing	$J\,mol^{-1}$
G^o	Gibbs free energy	$J\,mol^{-1}$
G_x	Soil heat flux measured by a flux plate	$W\,m^{-2}$
G_{total}	Combined measured soil heat flux	$W\,m^{-2}$
G_{stor}	Change in stored soil heat	$W\,m^{-2}$
h	Height	m
h	Planck's constant	erg s
h	Distance below the water table	m
h	Distance	m
h	Suction	$J\,m^{-3}$
^2H	Deuterium	—
^3H	Tritium	—
H	Hurst exponent	unitless
H	Sensible heat flux	$W\,m^{-2}$
H	Hydraulic head	m
H	Minimum distance between two particle surfaces or spheres	m
h_c	Height of fluid column	m
h_f	Pressure drop	m
H_{im}	Heat of immersion	$J\,m^{-2}$
h_m	Matric head of soil water	m
h_o	Osmotic pressure head of soil water	m
h_p	Pressure head	m
h_v	Velocity head	m
i	Infiltration rate	$m\,s^{-1}$
i	Index	—
i	Area for the solid–water interface	m^2
i/n_{io}	Concentration of ionic species in outer solution	$mol\,m^{-2}$
I	Cumulative infiltration	m
I	Intensity of light	$W\,m^{-2}$
I	Strength of applied electric field	$N\,C^{-1}$
I	Ionic strength	$eq\,L^{-1}$
I	Electric current	$J\,C^{-1}$
Ie'	Force acting on charge e'	N or $Kg\,m\,s^{-2}$
I_f	Final infiltration rate	$m\,s^{-1}$
I_o	Initial infiltration rate ($T = 0$)	$m\,s^{-1}$
I_z	z^{th} ionization potential	—
j	Index	—
J	Mass flux	$Kg\,m^{-2}\,s^{-1}$
k	Thermal gradient ratio	—
k	Boltzmann constant	$1.3805*10^{-23}$ $J\,K^{-1}\,mol^{-1}$
k	Intrinsic permeability	m^2
k	Stokes's constant	—
k_a	Air permeability in soil	$m^2\,s^{-1}$
ΔK	Change in kinetic energy	J
K	Equilibrium composition	—
$K(\theta)$	Unsaturated hydraulic conductivity	$m\,s^{-1}$
K	Proportionality constant	$8.988*10^9\,N\,m^2\,C^{-2}$
K	Constant	$8*10^{-36}\,J^2\,L^{-1}$ or $8*10^{-22}$ $ergs^2\,L^{-1}$
K	Hydraulic Conductivity	$m^2\,s^{-1}$
K_s	Saturated hydraulic conductivity	$m^2\,s^{-1}$
K_1	Thermodynamic equilibrium constant (initial condition)	unitless
K_2	Thermodynamic equilibrium constant (final condition)	unitless
K_a	Acid dissociation constant	unitless
K_{app}	Apparent acid dissociation constant	unitless
K_b	Base dissociation constant	unitless
K_c	Thermodynamic equilibrium constant	unitless

(continued)

Symbol	Description	Units
K_c	Exchange equilibrium constant	unitless
K^G	Gapon constant	unitless
K_h	Eddy diffusivity of heat	$m^2\,s^{-1}$
K_H	Henry's law constant	—
k_i^o	Standard partial molar compressibility	—
K_d	Dissociation constant	$mg\,kg^{-1}$
K_e	Effective hydraulic conductivity	$m\,s^{-1}$
K_j	Hydraulic conductivity of each layer	$m\,s^{-1}$
K_{oc}	Linear adsorption coefficient	$mL\,kg^{-1}$
K_{ow}	Octanol–water partition coefficient	unitless
K_p	Thermodynamic equilibrium constant (partial pressure)	unitless
K_s	Salting parameter	unitless
K_v	Eddy diffusivity of vapor	$m^2\,s^{-1}$
K_w	Autodissociation constant of water	unitless
k'	Pore shape factor	—
l	Tube length	m
l	Thickness of clay particle	m
l	Actual length of pore	m
l_e	Effective length of pore	m
L	Point of measurement within the hydrometer	—
L	Length	m
L_f	Distance from soil surface to wetting front	m
L_m	Latent heat of melting	$J\,kg^{-1}\,K^{-1}$
L_s	Latent heat of sublimation	$J\,kg^{-1}\,K^{-1}$
L_v	Latent heat of vaporization	$J\,kg^{-1}\,K^{-1}$
L_e	Effective length	m
m	Molecular weight	$g\,mol^{-1}$
m	Mass of electron	kg
m	Mass	kg
M	Mass of sample	kg
M	Molecular weight of water (mole basis)	$18\,g\,mol^{-1}$
M	Molar concentration	$mol\,L^{-1}$
M	Molar mass of water	g
m_a	Mass of pycnometer filled with air	kg
m_{an}	Molality of the anion	$mol\,g^{-1}$
m_{cat}	Molality of the cation	$mol\,g^{-1}$
m_d	Molecular weight of dry air	$g\,mol^{-1}$
m_i	Mass or number of moles of the various constituents in system	g or mol
m_v	Molecular weight of water	g
ΔM	Change in mass	kg
M_i	Mole fraction of water in inner solution	$mol\,mol^{-1}$
M_i	Mole fraction of the ith constituent	$mol\,mol^{-1}$
$M_{(i)}$	Mass of gas i	kg
M_n	Number of capillaries of radius R_n	unitless
M_o	Mole fraction in the outer solution	$mol\,mol^{-1}$
m_s	Mass of oven dry porous media sample	kg
M_s	Mass of solid	kg
m_{sw}	Mass of pycnometer filled with soil and water	kg
m_w	Mass of pycnometer filled with water	kg
n	Statistical number of clay layers	unitless
n	Electrolyte concentration in solution	$mol\,L^{-1}$ or ions cm^{-3}
n	Integer value	—
^{15}N	Nitrogen 15	—
N	Number of electrons in outermost shell	unitless
N	Number of Carbon 14 atoms	unitless
n^+	Cation concentration	$mol\,L^{-1}$
n^-	Anion concentration	$mol\,L^{-1}$
N_A	Avogadro's number	$6.023*10^{23}$

Symbol	Description	Units
n_i	Molarity of solute i	mol L^{-1}
n_i	Concentration solutes in inner solution	mol cm^{-3}
n_o	Concentration of ionic species in outer solution	ions cm^{-3}
N_{pv}	Number of pore volumes	—
^{18}O	Oxygen 18	—
p	Vector	unitless
P	Pressure	Pa or Nm^{-2}
p	Potential energy	J kg^{-1}
^{32}P	Phosphorus 32	—
P	Energy consumed in photosynthesis	W m^{-2}
P	Net pressure	Pa
P	Photosynthesis	W m^{-2}
P	Pressure	J m^{-3}
P	Precipitation rate	m s^{-1}
P	Percent media remaining in suspension at reference point	unitless
P	Gas pressure	Pa or N m^{-2}
$\mathscr{P}$	Peclet number	unitless
p_0	Pressure of chamber 2 after isothermic expansion	Pa or N m^{-2}
P_0	Gas pressure required for monolayer saturation at experimental temperature	Pa or N m^{-2}
p_1	Initial pressure	Pa or N m^{-2}
p_2	Final pressure (chamber 1)	Pa or N m^{-2}
Pa	Atmospheric pressure	Pa or N m^{-2}
P_a	Air pressure	Pa or N m^{-2}
P_d	Outer pressure	Pa or N m^{-2}
P_e	Envelope pressure	Pa or N m^{-2}
P_e	External pressure	Pa or N m^{-2}
P_h	Pressure due to standing head in soil column	Pa or N m^{-2}
P_i	Internal force between particles	N or Kg m s^{-2}
p_m	Pressure equal to the matric potential of soil	Pa or N m^{-2}
p_o	Potential energy at reference state	J kg^{-1}
p_o	Vapor pressure at a free surface of pure water under atmospheric conditions	Pa or N m^{-2}
P^o	Standard-state pressure	Pa or N m^{-2}
P_r	Precipitation rate	m s^{-1}
P_s	Swelling pressure	Pa or N m^{-2}
p_t	Tensiometer pressure	Pa or N m^{-2}
P_w	Pore-water pressure	Pa or N m^{-2}
q	Specific humidity	—
q	Flux density (specific discharge)	m s^{-1}
q	Number of atoms per unit of substance making up the particles	unitless
q	Darcy flux	m s^{-1}
q/q'	Electrical charges	C (coulombs)
q_n	Hydraulic flow rate	m^3 s^{-1}
Q	Rate of volume flow	m^3 s^{-1}
Q	Actual composition	—
Q_c	Electric charge	C
Q_e	Dry-weight concentration	mg kg^{-1}
Q_h	Heat of adsorption	cal mol^{-1}
Q_h	Heat given off due to intermolecular potential	cal g^{-1}
Q_{h1}	Heat of adsorption of first layer of adsorbing gas	cal mol^{-1}
Q_{h2}	Heat of gas liquification	cal mol^{-1}
Q_i	Flux density of gas	m s^{-1}
Q_o	Total outflow	m^3 s^{-1}
Q_{10}	Respiration rate factor	—
r	Reflectivity	unitless
r	Tube radius; particle radius	m

(continued)

Symbol	Description	Units
r	Distance between two atoms	m
R	Retardation coefficient	dimensionless
R	Hydraulic resistance	$Pa\,s\,m^{-1}$
R	Universal gas constant	$8.314\,J\,K^{-1}\,mol^{-1}$
R	Runoff rate	$m^3\,s^{-1}$
R_1, R_2	Radii of curvature of meniscus	m
R_a	Corrected hydrometer reading	—
Re	Reynolds's number	dimensionless
R_e	Hydraulic resistence of soil crust	$m\,s^{-1}$
R_n	Net radiation	$W\,m^{-2}$
R_r	Radius of rotation	m
R_{r2}	Depth to point of measurement	m
R_s	Hydraulic resistence of soil subcrust	$m\,s^{-1}$
R_v	Ideal gas constant of vapor	$J\,Kg^{-1}\,K^{-1}$
r_b	Respiration base rate	$mg\,m^{-3}\,s^{-1}$
r_c	Stomatal resistance or areal resistance	$s\,m^{-1}$
r_e	Effective pore radius	m
r_s	Charge separation distance	m
s	Slope of the saturation vapor curve	—
s	Gas constituent	—
s	Degree of saturation	—
s	Specific surface area of soil	$m^2\,kg^{-1}$
s	Triangular volume of the tube	m^3
^{34}S	Sulfur 34	—
S	Solute adsorbed	$\mu g\,g^{-1}$
S	Slope	—
S	Source or sink term of energy or gas	—
S	R/a; reduces to $2 + H/a$	dimensionless
S	Sorption	$m^3\,g^{-1}$
S	Siemens	$A\,V^{-1}$
S	Entropy	$J\,mol^{-1}\,K^{-1}$
s_b	Specific surface area per unit bulk volume of soil	$m^2\,kg^{-1}$
S_e	Effective saturation	decimal
S_i	Concentration of sorbed chemical (i^{th} species)	$mol\,kg^{-1}$
S_{id}	Saturation index	—
s_k	Geometrical shape factor of pore space	—
s_m	Specific surface area per unit mass	$m^2\,kg^{-1}$
SMOW	Standard mean ocean water	—
s_o	Reference distance	m
S^o	Molar solubility	$mg\,L^{-1}$
S_s	Specific storage	m^3
S_{sat}	Soil sorbed contaminant concentration	$\mu g\,kg^{-1}$
S_t	Stored energy	J
s_v	Specific surface area per unit volume	$m^2\,kg^{-1}$
t	Time	s
T	Temperature	K
T_f	Final temperature	K
T	Half life $(\ln 2/\lambda)$	—
T_{DP}	Dew point temperature	°C
T_m	Melting point	K
T_o	Initial or surface temperature	°C
t_p	Time of ponding	s
T_w	Wet-bulb temperature	°C
TU	Tritium units	—
u	Velocity parallel to mean flow	$m\,s^{-1}$
ΔU	Change in potential energy	J
U	Velocity	$m\,s^{-1}$
U	External electric field	N

(*continued*)

Symbol	Description	Units
v	Velocity normal to mean flow	m s^{-1}
v	Velocity of soil particle	m s^{-1}
v	Linear velocity	m s^{-1}
v	Velocity	m s^{-1}
V	Volume of water in the meniscus	m^3
V	Volume	m^3
V	Molar volume of water	$18\ \text{cm}^3\ \text{mol}^{-1}$
v_1, v_2	Constant velocity	m s^{-1}
V_1	Volume of compression chamber 1	m^3
V_2	Volume of compression chamber 2	m^3
V_a	Volume of air	m^3
V_A	Average attraction force per unit of material	ergs cm^{-2}
v_e	Effective velocity	m s^{-1}
V_E	Volume rate of flow	$\text{m}^3\ \text{s}^{-1}$
V_f	Viscous force	N
V_f	Fluid volume	m^3
V_i	Partial molar volumes of i under actual conditions	m^3
V_i^o	Partial molar volumes of i under standard conditions	m^3
V_l	Liquid volume	m^3
V_m	Volume of adsorbed gas forming monomolecular layer on adsorbant	m^3
v_o	London frequency	s^{-1}
V_r	Potential energy due to repulsive forces	J m^{-3}
V_R	Repulsive force for energy between two spherical particles of equal dimension	ergs cm^{-1}
$V_R^{\psi\infty}$	Repulsive energy between particles at constant potential	J m^{-2}
V_s	Solid volume	m^3
V_t	Total volume	m^3
V_T	Total potential energy	J m^{-2}
v_w	Partial volume of water	m^3
V_w	Partial molal volume of soil water	m^3
V_w	Volume of water in soil pores	m^3
w	Velocity in the vertical direction	m s^{-1}
w	Mass wetness	g
W	Work	J
W	Work done by the system	J
W	Weight of oven-dried soil	g
W	Hydration energy	J m^{-2}
W^σ	Reduced repulsive energy at constant charge density	J m^{-2}
W^ψ	Reduced repulsive energy of two parallel flat plates of constant potential	J m^{-2}
W_m	Mass of water sorbed to provide one molecular thickness of coverage	g
w_s	Specific volume of water	m^3
x	Distance	m
X	Concentration of each ion in solution	ions cm^{-3}
X	Negatively charged surface (assuming an adequate number of exchange sites are present)	unitless
x_d	Equals d; distance in equation 3.70	m
x_i	Mole fraction of ith ions	ions mol^{-1}
y	Distance from reference	m
Y	Concentration of each electrolyte between plates	mol L^{-1}
z	Distance	m
z	Depth relative to soil surface	m
z	Valence of ion	unitless
z	Gravitational potential	J m^{-3}
Z_D	Damping depth	m
Z	Depth of soil column or water-table depth	m

Symbol	Description	Units
Z	Concentration of dissociated and absorbed cations	ions cm^{-3}
$Z(p)$	Compressibility factor	unitless
z_{soil}	Depth to soil reference point	m
z_{wt}	Depth to water table	m
Z_x	Increment in distance from standard height of point X	m
∇	Laplacian differential operator	unitless
Å	Angstrom	10^{-10} m
∂q	Increment of heat evolved	W m^{-2}
∂_w	Increment of water added	m^3
α	A constant	unitless
α	Dispersivity	cm
α	Aquifer compressibility	unitless
α	Polarizability	m^3
β	Bowen ratio, soil specific constant, fluid compressibility	dimensionless
γ	Psychrometric constant	dimensionless
γ	Temperature change with depth	°C m^{-1}
γ	Surface tension	ergs cm^{-2} or dynes cm^{-1} or J m^{-2} or N m^{-1}
γ_T	Thermodynamic psychrometric constant	unitless
γ_i	Molar activity coefficient of species i	unitless
γ_{LV}	Surface tension of the fluid	J m^{-2} or ergs cm^{-2}
γ_m	Surface tension or energy of the material	J m^{-2}
γ_{YL}	Interfacial tension	J m^{-2} or N m^{-1}
δ	Plate thickness at distance $2d$, depth	m
δ	Net increase of moles of gas in formation reaction	unitless
$\Delta\psi$	Potential difference	J m^{-3}
ΔG^o	Gibbs free energy of formation	kcal mol^{-1}
ΔG^s	Change in surface free energy	kcal mol^{-1}
ΔH	Quantity of heat	J
ΔH_f	Heat of fusion	kJ mol^{-1}
ΔH^o	Change of enthalpy	kJ mol^{-1}
ε	Ratio of molecular weights of water vapor and dry air	unitless
ε	Emissivity	unitless
ε	Dielectric constant (80 for water)	unitless
ε	Permittivity of the solution	esu^2 erg^{-1} cm^{-1}
ε_o	Permittivity constant of vacuum	8.854*10^{-12} Fm
ε_v	Velocity for a potiental gradient of 1 volt	m s^{-1}
ζ	Zeta potential	V
ζ	Temperature change with depth	K
ζ	Density	kg m^{-3}
η	Dynamic viscosity	Pa s^{-1}
η_e	Electrolyte concentration	ion cm^{-3}
θ	Angle	radians
θ	Volumetric water content	m^3 m^{-3}
θ_a	Volume fraction of air	m^3 m^{-3}
θ_i	Initial volumetric water content	m^3 m^{-3}
θ_m	Maximum volumetric water content	m^3 m^{-3}
θ_o	Initial soil-water content	m^3 m^{-3}
θ_r	Residual saturation	m^3 m^{-3}
θ_s	Saturated water content	m^3 m^{-3}
θ_t	Volumetric water content of transition zone during infiltration	m^3 m^{-3}
θ_x	Volumetric water content at distance x	m^3 m^{-3}
$\cos\theta$	Contact angle	degrees
κ	Debye	m^{-1}
κ^{-1}	Effective thickness	m
κ_c	Thermal conductivity	W m^{-1} K^{-1}

(continued)

Symbol	Description	Units
κ_i	Thermal conductivity of a soil constituent	$\text{W m}^{-1}\,\text{K}^{-1}$
$\kappa(t)$	Soil-water suction function	J m^3
λ	Wavelength	nm
λ	Pore size distribution index	—
λ	Decay constant for Carbon 14	$1.2*10^{-4}\ \text{yr}^{-1}$
λ	Wavelength of London frequency	nm
λ'	Specific or electrical conductivity of water	S m^{-1} or ohms cm^{-1}
μ	Rate constant	s^{-1}
μ	Potential energy of medium under consideration	J kg^{-1}
μ_f	Chemical potential at a distance 1/2 particle diameter from solid surface	J kg^{-1}
μ_i	Chemical potential of the ith component	J kg^{-1}
μ_i	Chemical potential between two plates	J kg^{-1}
μ_o	Chemical potential	J kg^{-1}
μ_v	Chemical potential of water (vapor phase)	J kg^{-1}
μ_w	Chemical potential for the soil-water component	J kg^{-1}
$\mu_w(m_i)$	Chemical potential of soil water	J kg^{-1}
μ_w^o	Chemical potential for pure free water at the same temperature and atmospheric pressure	J kg^{-1}
μ_y	Chemical potential at point y	J kg^{-1}
ν	Kinematic viscosity	$\text{m}^2\,\text{s}^{-1}$
ν_a	Viscosity of air	$\text{m}^2\,\text{s}^{-1}$
ν_d	Volume drained (fluid)	m^3
Π	Osmotic pressure of solution	J kg^{-1}
Π	Period of time	T
Π	Osmotic potential of solution	J kg^{-1}
π	3.14159...	unitless
π_i	Osmotic coefficient of solute i	J kg^{-1}
ρ	Reaction constant	—
ρ	Density of solution	kg m^{-3}
ρ_a	Dry air density	kg m^{-3}
ρ_α	Density of phase	kg m^{-3}
ρ_f	Fluid density	kg m^{-3}
ρ_g	Gas phase density	kg m^{-3}
ρ_s	Density of solid (particle) phase	kg m^{-3}
ρ_{vapor}	Density of vapor	kg m^{-3}
ρ_v	Water vapor density	kg m^{-3}
ρ_w	Density of water	kg m^{-3}
ρ_b	Bulk density	kg m^{-3}
σ	Stefan–Boltzmann constant	$10^{-8}\text{Wm}^{-2}\text{K}^{-4}$
σ	Charge on colloid surface	esu cm^{-2} or meq cm^{-2}
σ	Adsorption per unit area	mol m^{-2}
σ	Substituent constant	—
σ	Surface energy per unit area of fluid–gas interface	J m^{-2}
σ^+	Charge due to a surplus of cations	C m^{-2}
σ^-	Charge due to a deficiency of cations	V
σ_g	Intergranular pressure	Pa
σ_r	Modulus of rupture	dynes cm^{-2}
σ_s	Soil stress	N
σ_s	Limiting stress	N
σ_T	Total charge	V
τ	Tortuosity	m m^{-1}
τ	Transmissivity	$\text{m}^2\,\text{s}^{-1}$
τ	Tensile stress normal to the plane of the crack	N
$\tau_{1/2}$	Half-life	s
τ_f	Internal frictional force	N

(continued)

Symbol	Description	Units
ε	Geometric term	m^3
ϕ	Polar coordinate	unitless
ϕ	Phase shift	unitless
ϕ	Porosity	unitless
ϕ_α	Volume fraction of considered phase	unitless
ϕ_a	Volume fraction of air in soil pores	unitless
ϕ_e	Effective porosity	unitless
ϕ_g	Volume fraction of gas in soil pores	unitless
ϕ_i	Gain or loss via various source/sink(s)	$kg\,m^{-3}\,s^{-1}$
ϕ_l	Volume of water in media pores	unitless
ϕ_r	Residual water content	unitless
ϕ_s	Volume of solids in soil pores	unitless
ϕ_t	Total volume of soil	unitless
ϕ_v	Volume of all void spaces	unitless
ψ	Effective suction at wetting front	$J\,m^{-3}$
ψ_b	Brooks–Corey bubbling pressure	$J\,m^{-3}$
ψ_o	Saturated suction (matric suction = 0)	$J\,m^{-3}$
ψ	Water potential	$J\,m^{-3}$
ψ	Electrical potential of the colloid at specified distance	V
ψ	Surface charge	V
ψ_a	Pneumatic (air or vapor) potential	$J\,m^{-3}$
ψ_b	Overburden pressure potential	$J\,m^{-3}$
ψ_c	The matric potential for which $K = 1/2K_s$	$J\,m^{-3}$
ψ_d	Electric potential between layers	V
ψ_g	Gravitational potential	$J\,m^{-3}$
ψ_h	Hydrostatic pressure potential	$J\,m^{-3}$
ψ_i	Partial potential	$J\,m^{-3}$
ψ_m	Matric potential	$J\,m^{-3}$
ψ_{mL}	Matric potential of the loaded sample	$J\,m^{-3}$
ψ_{mu}	Matric potential of the unloaded sample	$J\,m^{-3}$
ψ_o	Osmotic potential	$J\,m^{-3}$
ψ_o	Electric potential at surface	$ergs\,esu^{-1}$
ψ_p	Pressure potential	$J\,m^{-3}$
ψ_r	Potential at pore neck (r in meters)	$J\,m^{-3}$
ψ_R	Potential in middle of pore (R in meters)	$J\,m^{-3}$
ψ_t	Total potential	$J\,m^{-3}$
ψ_w	Value of the matric potential when soil stress and external air pressure are zero	$J\,m^{-3}$
ω	Angular frequency	$radians\,s^{-1}$
ω	Angular velocity	$degrees\,s^{-1}$

Note: Units in the symbols list are given as equivalent or derived SI units. However, these units may not be exactly the same as the units for the specified symbol within the text. It should also be noted that we have used $J\,m^{-3}$ for most of our potentials. This is easier to use unless one is concerned with the specific water density being investigated within the unsaturated zone. See pp. 587–595 for further assistance concerning units.