Unsaturated Zone Hydrology for Scientists and Engineers

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Principles of Water Flow in Soil

INTRODUCTION

By definition, a fluid is a substance that is capable of flow. In soils, this includes both the liquid and gas phases. In the gas phase molecules are spaced farther apart, while in the liquid phase, molecules are more closely bound. The intermolecular cohesive forces are considerably smaller in a gas because of the separated distance of molecules, compared to those of a liquid. Normally, fluids always possess elastic properties while under compression due to their inability to resist shear stress. Consequently, fluids can alter their shape and flow characteristics, depending on the physical and chemical characteristics of the medium.

Physically, fluid characteristics may be expressed as density, specific gravity, specific volume, and specific weight. Density is the mass of the fluid per unit volume, specific gravity is the

TABLE 6.1 Physical Properties of Water (Liquid Phase)

Temperature (°C)	Density, ρ ($g\ cm^{-3}$)	Heat capacity, C_p ($J\ g^{-1}\ K^{-1}$)[†]	Surface tension, γ ($10^{-3}\ N\ m^{-1}$)	Thermal conductivity ($10^{-3}\ W\ K^{-1}\ m^{-1}$)[‡]	Viscosity, η (Pa s)[§]
0	0.99984	4.2161	75.6	561.0	0.001793
4	1.00000	4.2077	75.0	569.4	0.001567
5	0.99999	4.2035	74.8	573.6	0.001519
10	0.99970	4.1910	74.2	586.2	0.001307
15	0.99913	4.1868	73.4	594.5	0.001139
20	0.99821	4.1826	72.7	602.9	0.001002
25	0.99708	4.1784	71.9	611.3	0.000890
30	0.99565	4.1784	71.1	619.6	0.000798
35	0.99406	4.1784	70.3	628.0	0.000719
40	0.99222	4.1784	69.5	632.2	0.000653
45	0.99024	4.1784	68.7	640.6	0.000596
50	0.98803	4.1826	67.9	644.8	0.000547
60	0.98320	4.1843	66.2	654.3	0.000466
70	0.97778	4.1895	64.5	663.1	0.000404
80	0.97182	4.1963	62.7	670.0	0.000354

Source: Data compiled from Lide (1992)
[†] To convert to ($cal\ g^{-1}\ deg^{-1}$), divide by 4.1868
[‡] To convert to ($cal\ cm^{-1}\ sec^{-1}\ deg^{-1}$) $\times 10^{-3}$, divide by 418.68
[§] Dynamic viscosity: to convert to ($g\ cm^{-1}\ sec^{-1}$) $\times 10^{-2}$, divide by 1000. To obtain kinematic viscosity, divide dynamic viscosity by fluid density.

ratio of fluid density to the density of pure water (which is dimensionless), specific volume is the volume per unit mass, and specific weight is the weight per unit volume of the fluid. Some of the physical properties of water are given in table 6.1. As shown in this table, the density, surface tension, and viscosity decrease with an increase in temperature.

The following is assumed of a perfect fluid: it lacks viscosity, it is resistance free, it is incompressible (i.e., it has constant density), and it has no irrotational flow. Fluid flow is irrotational when there is no angular momentum of the fluid about any object or point, that is, a small wheel with a fixed rotational point in the center of its mass will not rotate about that point if the wheel is placed (or submerged) in the fluid's path. The assumption of these characteristics implies that there will be no friction between various layers in the fluid as well as no friction between the fluid and the boundary wall. As a result, the fluid is like an aggregation of small particles that support pressure normal to the particle surface, but glide over other particles without resistance.

6.1 BERNOULLI'S EQUATION

The process of fluid transport always obeys the law of conservation of matter and energy. Simply defined, inflow = outflow ± change in storage. This principle applies equally to flow into a lake, the cross-section of an aquifer, or a specific volume of soil. Mathematical formulations of transport processes through soil must reflect the law of conservation, but because soil-water flow problems are generally considered to be isothermal, the law of conservation of energy can be omitted; however, the law of conservation of matter cannot. This law is expressed in the continuity equation, which for one-dimensional flow is $\partial\theta/\partial t = -\partial q/\partial x$, where θ is the volume fraction of water.

In figure 6.1, we examine fluid flow through a pipe, labeled A at one end and B at the other. For a specific time interval Δt, the fluid at point A (cross-sectional area of pipe) moves a distance $\Delta x_1 = v_1 \Delta t$. Considering cross-sectional area A, the mass within this area is $\Delta m_1 = \rho_1 A \Delta x_1 = \rho_1 A v_1 \Delta t$, where p and v refer to pressure and velocity; likewise, the mass of fluid moving through the upper end of the pipe may be expressed as $\Delta m_2 = \rho_2 B \Delta x_2 = \rho_2 A v_2 \Delta t$. Since the flow is steady and mass is conserved, the mass which crosses A during time Δt must also equal the mass which crosses B during this same interval, resulting in $\Delta m_1 = \Delta m_2$ and $\rho_1 A v_1 = \rho_2 B v_2$; this expression is termed the equation of continuity. The equation of continuity simply implies that the product of the area and the velocity of the fluid at all points through the pipe is constant—therefore, the pipe does not have to be the same diameter at each end.

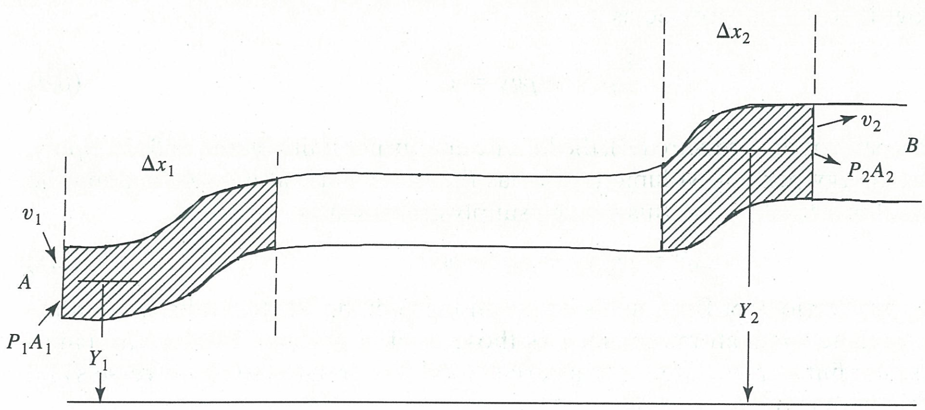


Figure 6.1 Incompressible fluid flowing through a constricted pipe (steady flow). The fluid in the cross-sectional length, Δx_1, moves to section Δx_2. The volume is equal in the two sections. Y is height above reference, v is velocity, x is length, and P is pressure. A and B are cross-sectional areas at each end of the pipe.

Assuming that fluid moves at a steady rate through a pipe of varying cross-sectional area and that the pipe varies in elevation, the fluid pressure along this pipe will change. In 1738, the Swiss physicist Daniel Bernoulli (1700–1782) derived an expression relating pressure to fluid velocity, elevation, and conservation of energy, when applied to a perfect fluid. The state of a fluid at any point may be characterized by the following four quantities: **(1)** pressure, P; **(2)** velocity, v; **(3)** elevation, h; and **(4)** density, ρ.

Consider a pipe with varying elevation and cross sectioned area, as in figure 6.1; flow velocity is nonuniform at any point of interest, Δx. The force, F, at the lowest end of the pipe in the figure is $P_1 A_1$, where P is pressure and A is cross-sectional area. The work done by the represented force is $W_1 = F_1 \Delta x_1 = P_1 A_1 \Delta_1 = P_1 \Delta V$, where ΔV represents the volume of the lower cross-hatched region. The work done at the upper end of the pipe would follow the same sequence (replacing all subscripts 1 by subscripts 2), except that the fluid force would be negative since the fluid force opposes the displacement, that is, the sign would be negative. In addition, the fluid volume passing through 1 in time Δt is equal to the fluid volume passing through 2 during the same Δt.

As a result, the net work performed by these forces during time Δt may be expressed as

$$W = (P_1 - P_2)\,\Delta V \tag{6.1}$$

The work will be divided into two parts: one part that changes the kinetic energy of the fluid and another part that changes the gravitational potential energy. Assuming that Δm is the mass passing through the pipe during Δt, the change in kinetic energy, ΔK, can be expressed as

$$\Delta K = \frac{1}{2}(\Delta m)v_2^2 - \frac{1}{2}(\Delta m)v_1^2 \tag{6.2}$$

The change in potential energy may be written as

$$\Delta U = \Delta mgy_2 - \Delta mgy_1 \tag{6.3}$$

where g is the acceleration of gravity. By applying the work–energy theorem ($W = \Delta K + \Delta U$) to the fluid volume we obtain

$$(P_1 - P_2)\Delta V = \frac{1}{2}(\Delta m)v_2^2 - \frac{1}{2}(\Delta m)v_1^2 + \Delta mgy_2 - \Delta mgy_1 \tag{6.4}$$

Since $\rho = \Delta m/\Delta V$, one may divide by ΔV; with some rearrangement, equation 6.4 reduces to

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho gy_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho gy_2 \tag{6.5}$$

Equation 6.5 is Bernoulli's equation as typically applied to a nonviscous, incompressible fluid during steady flow. It is often expressed as

$$P + \frac{1}{2}\rho v^2 + \rho gy = C \tag{6.6}$$

This states that the sum of the pressure (P), the kinetic energy per unit volume of fluid ($\frac{1}{2}\rho v^2$), and the potential energy per unit volume (ρgy), has the same value at all points along the streamline of flow. For a fluid at rest, this may be simply expressed as

$$\Delta P = \rho g(y_2 - y_1) = \rho gh \tag{6.7}$$

Some interesting applications of Bernoulli's equation include: the Venturi tube; streamline flow around an airplane wing; atomizers such as those used in perfume bottles and paint-sprayers; and vascular flutter associated with arterial blood flow (related to blood vessels and heart valves within humans).

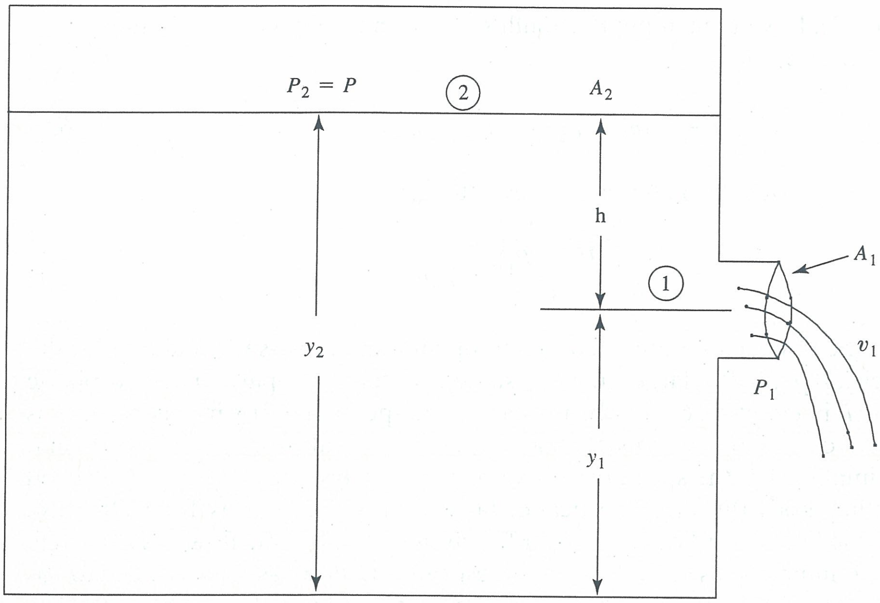


Figure 6.2 Efflux, v_1, from hole in side of container; $v_1 = (2gh)^{1/2}$. A is cross-sectional area of the exit hole (subscript 1) and container (subscript 2), y is fluid level to efflux hole (subscript 1) and container (subscript 2), P is pressure in container (point 2), and P_1 is atmospheric pressure or pressure at outlet (point 1).

QUESTION 6.1

A large tank is filled with water; it develops a hole in its side 20 m below the water level (shown in figure 6.2). If the rate of flow from the hole is 4.2×10^{-5} m^3/s: **(a)** What is the speed at which water leaves the hole? **(b)** What is the hole diameter?

QUESTION 6.2

Geyser surges result from water becoming superheated and flashing to steam as pressure is first released. Using our imagination to idealize the famous geyser "Old Faithful" in Yellowstone National Park in Wyoming as a steady water spout, we can apply Bernoulli's equation to determine the velocity of the water as the geyser erupts, as well as determining the pressure in the heated chamber below ground. The height of the eruption typically reaches 40 m above ground surface. **(a)** What is the velocity of the water as it leaves the ground? **(b)** What is the pressure (above atmospheric) in the heated underground chamber?

6.2 TORRICELLI'S LAW

Because all fluids have mass, an unbalanced force that acts on the particles of a fluid will cause an acceleration of those particles, according to Newton's law of motion. An example of this can be seen in any water-supply system; this is especially true of those systems that use large tanks for storage. Such tanks can either be pressurized, or be placed at a higher elevation than the discharge point (e.g., a gravity system). Figure 6.2 shows a tank containing a fluid of density ρ with a hole in its side at distance y_1 from the bottom; here we assume the tank is not open to the atmosphere, so that the air space above the fluid level in the tank is maintained at a pressure P. By also assuming that the cross-sectional area of the tank is large compared to the cross-sectional area of the hole in its side ($A_2 >> A_1$), the fluid will be at rest at the top (point 2). If we then apply Bernoulli's equation to points 1 and 2, and noting that $P = P_1$ at the hole, we obtain

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \rho g y_2 \tag{6.8}$$

however, since $y_2 - y_1 = h$, equation 6.8 can be rewritten as

$$v_1 = \sqrt{\frac{2(P_2 - P_1)}{\rho} + 2gh} \tag{6.9}$$

The flow rate from the hole can be obtained by multiplying the cross-sectional area of the hole times its velocity, $A_1 v_1$. If P is large (in a pressurized system) compared to atmospheric pressure, the term $2gh$ in equation 6.9 can be neglected; the speed of efflux in this instance is primarily a function of P. For systems that are open to the atmosphere, $P_2 = P_1$ and $v_1 = (2gh)^{1/2}$. This implies that the speed of efflux for the open system is equal to the speed gained by a free-falling body through a vertical distance h; this is known as Torricelli's law.

Flow through the hole described above is different than if that same flow were through a pipe, because the confining walls of the pipe offer resistance to flow (assuming the fluid has viscosity). Because of viscosity, head losses occur in real systems due to frictional losses. The magnitude of these frictional losses depends upon whether flow is laminar or turbulent as defined by the Reynold's number. The flow system in figure 6.3 shows three equally spaced manometers at points C_1–C_3 along a pipe. The reservoir is held at a constant level and velocity of flow through the pipe is controlled by the valve at point C_4, such that constant pressure and steady flow are maintained. If the valve is closed, the fluid level will be equal in all manometers (point AB), hence h (AC_1) will indicate an equal pressure at all points C_1–C_3 along the pipe. However, once the valve C_4 is opened to a certain setting, steady flow will be achieved through the pipe and the height of fluid in each manometer will be at different levels, as indicated by points 1, 2, and 3. The greater the flow (i.e., the more open the valve), the greater the drop of the manometers wll be. As discussed in chapter 4, we know that the height of water in each manometer is a measurement of the pressure at points C_1–C_3 in the pipe. Because the system is at constant pressure and steady flow, the straight line along points 1–3 in figure 6.3 indicates a uniform pressure drop in the manometers. Now recall that the confining walls of the pipe offer resistance to flow, thus, the drop in pressure indicated by h_f in figure 6.3 is due to fluid friction, termed the friction head. For varying flow velocities

$$h_f = Kv^2 \tag{6.10}$$

where K is the proportionality constant (to be discussed in chapter 7) and v is fluid velocity in the pipe (m s^{-1}). Equation 6.10 depends on the Reynolds number (Re); for Re $< 2{,}000$, $h_f = Kv$.

Figure 6.3 shows an immediate drop in fluid level from point A of the reservoir to point 1 in the first manometer, which is indicated by h_v. Since equation 6.10 implies that friction head is proportional to the velocity squared, then from Torricelli's law, the drop in potential energy is due to the drop in fluid level from A to C_1. This drop must be converted into kinetic energy in the fluid, and since $v_1 = (2gh)^{1/2}$, then

$$h_v = \frac{v^2}{2g} \tag{6.11}$$

where h_v is the distance from point A to point 1, termed the velocity head in units of length (m). The pressure at any point from C_1–C_3 is determined from the height of fluid in the

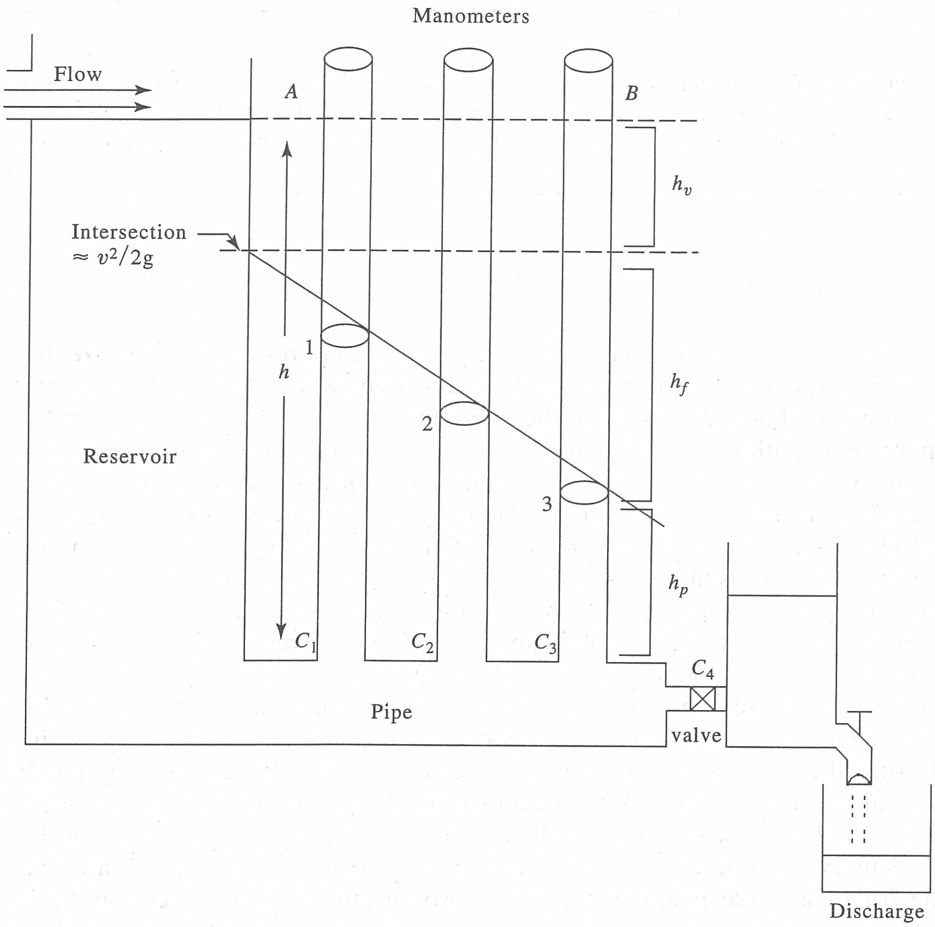


Figure 6.3 Illustration of manometer levels attached to pipe exiting reservoir. Points 1, 2, and 3 represent fluid levels in manometers located at points C_1, C_2, and C_3, while h_v, h_f, h_p refer to velocity head, pressure drop, and pressure head at point B; h is fluid level at point A.

manometer at the respective point such that

$$h_p = h - (h_v + h_f) \tag{6.12}$$

where h_p is the pressure head (m) and h is total head (m). An important point to remember here is that when fluid velocity increases, pressure decreases; and as fluid velocity decreases, pressure increases. In nature, an example of this would be when a wide, gently flowing river passes through a narrow canyon—the velocity of flow increases, but the pressure in the river decreases.

QUESTION 6.3

Calculate the pressure at a depth of 500 m beneath a lake's surface. Assume the density of water is 1.0×10^3 kg/m^3 and that atmospheric pressure, P_a, equals 1.01×10^5 Pa.

QUESTION 6.4

Water behind a dam of width w is filled to a height H. What is the resultant force on the dam? Give a general solution, not a numerical one.

QUESTION 6.5

Using figure 6.2 and the application of Bernoulli's equation to Torricelli's law (as given in equations 6.8 and 6.9), determine the velocity at which fluid will exit the small hole (point 1) on the right side of the tank when the fluid level is a distance h above the hole. In this case you are seeking a general solution, not a numerical one.

6.3 POISEUILLE'S LAW

Various models have been developed to investigate the effects of porosity and pore-size distribution on fluid flow. The best known is perhaps Poiseuille's law, which describes the laminar flow of a fluid in a small cylindrical tube. The tube radius in this case is that of a capillary tube, analogous to pore radius within soils and geologic material. Poiseuille's law states that the discharge rate, Q, of a fluid in a cylindrical tube of small, fixed radius, R, is dependent upon the driving force acting on the fluid as well as upon the internal friction forces between molecules within the fluid, characterized by the fluid viscosity, η. On a volume basis, the gradient of the hydraulic potential, $-dP_h/dx$, is the driving force, considered constant for this discussion.

Fluid viscosity may be explained with the aid of figure 6.4, which shows a fluid layer trapped between two parallel plates of solid substance. If one moves the lower plate at a constant velocity, v, relative to the upper plate, the velocity of the fluid at the boundary with the upper plate will be zero, but also equal to v at the boundary of the lower plate, because of adhesive forces. Assuming the steady velocity between the two plates, $v(y)$ will increase linearly with the distance y to the lower plate. The internal friction force per unit area within the fluid, τ_f, tends to retard the movement of the plate. To maintain the velocity, v, of the plate, the force per unit area applied to the plate must be equal and opposite to τ_f. The force is inversely proportional to the distance, h, between the two plates, also implying that the frictional force per unit area, τ_f, is proportional to the velocity gradient dv/dy. In this case, the proportionality factor is the fluid viscosity, η, which may be expressed such that

$$\tau_f = \eta \frac{dv}{dy} \tag{6.13}$$

As shown in figure 6.5, the rate of discharge from a small tube can be determined when the velocity distribution in the tube is known as a function of r. Since we assume laminar flow, the velocity of the fluid at the wall of the tube is zero, and has a maximum value in the center of the tube. Velocities from zero to maximum will depend on the radial distance from the center of the tube, r; equal velocities are present in concentric rings around the center. Thus, according to Poiseuille's law, the rate of discharge of a small cylindrical tube may be expressed as

$$Q = -\frac{\pi r^4}{8\eta}\left[\frac{dP_h}{dx}\right] \quad \text{or} \quad -\frac{\pi r^4}{8\eta}\left[\frac{\Delta P_h}{h}\right] \tag{6.14}$$

where Q is the rate of volume flow ($m^3\ s^{-1}$), η is the coefficient of viscosity of the fluid ($\mu Pa\ s^{-1}$), r is the tube radius (m), h is the tube length (m), and P is the pressure (Pa). Note here that $P = \rho gh$; often P is expressed as ΔH. If ΔH is used, the numerator on the right-hand side of the equation (within the brackets) must be multiplied by ρg. Normally, the pressure units would then be in the *cgs* system expressed as dynes cm^{-2}, and the resulting viscosity would have units of poise or dyne-sec cm^{-2}; all length units would be expressed in terms of cm rather than m. Equation 6.14 states that the rate of fluid flow through a cylindrical tube is

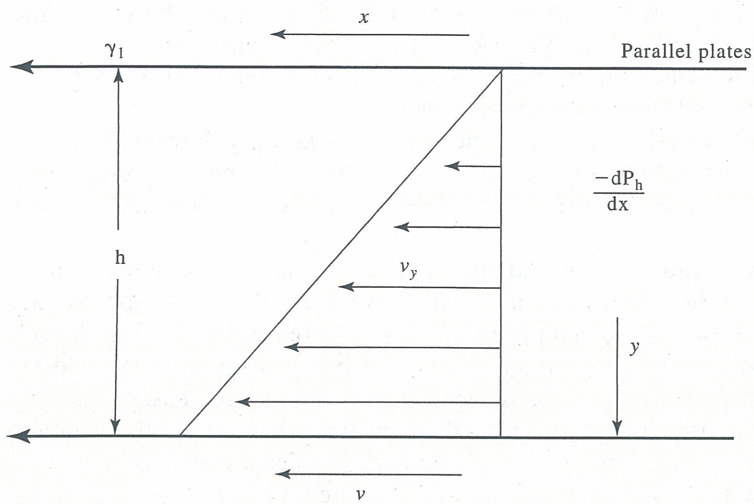


Figure 6.4 Flow between two parallel plates to illustrate viscosity, assuming a no-slip condition along plates. x is distance, v is velocity, γ_f is shear stress ($\gamma_f = \eta\, dv/dx$) exerted in direction x on fluid surface, y is distance of v_y above reference, and I is distance between plates (L).

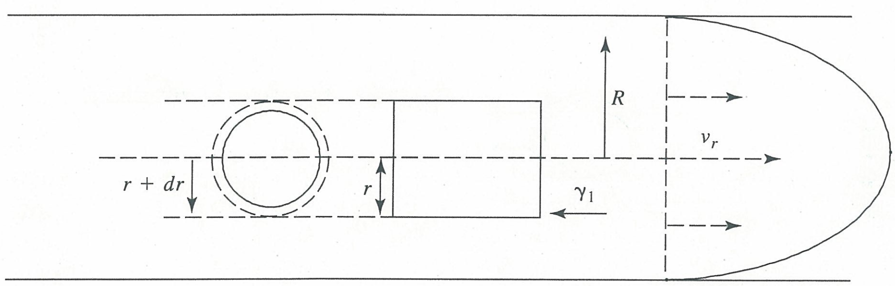


Figure 6.5 Laminar flow of a liquid in a tube illustrating Poiseuille's law. The rate of discharge can be calculated when the velocity distribution in the tube is known as a function of r. Due to friction between adjoining liquid layers and considering a liquid cylinder in the tube, r is radial distance from center, γ_1 is shear stress (as defined in figure 6.4), R is tube radius, and v_r is the velocity of fluid at radial distance r.

directly proportional to the fourth power of the radius of the tube and also to the pressure difference, but is inversely proportional to the viscosity of the fluid and the length of the tube. Poiseuille's law is true only when: **(1)** flow is steady and laminar; **(2)** the pressure is constant over every cross-section (no radial flow); and **(3)** fluid in contact with the tube wall is stationary.

QUESTION 6.6

An experiment to measure water flow is being conducted in a vertical capillary tube. At the bottom of the tube the water pressure, P_h (potential), is 7.0 kPa; at the top of the tube water pressure is 1.0 kPa. Assuming that $\eta = 10^{-6}$ kPa s, $l = 0.5$ m, and the diameter of the capillary tube is 3.6×10^{-4} m, what is the rate of flow of water, Q, through this capillary tube?

6.4 FLOW CHARACTERISTICS: LAMINAR AND TURBULENT FLOW

The movement of a fluid can be characterized as either laminar or turbulent. If each particle of the fluid flows along a smooth path and the paths of each particle do not cross each other, the flow is termed laminar. As a result, the velocity of the fluid at any point along its flow-path remains constant in time. However, above a certain critical speed, fluid flow becomes turbulent. Turbulent flow is an irregular flow often characterized by small whirlpool-like regions. A familiar example of this would be the flow of a stream around a rock which projects above the stream's surface. Upon close observation, one would see the small whirlpool-like regions and eddy irregularities around the rock.

The flow path taken by a fluid particle in laminar flow conditions is termed a streamline (see figure 6.6). For laminar flow, no two streamlines may cross each other; if they do, a fluid particle could move either way at the crossover point, and the flow would be termed turbulent.

Flow of water in soils and other geologic material can occur in three dimensions, as determined by the potential gradient of the system. If one assumes steady, one-dimensional laminar flow of a fluid (incompressible) along a solid plane surface, the velocity profile of this flow would be as illustrated in figure 6.7. Since the distance to the wall, x, is at a right-angle to the velocity, then at $x = 0$, the velocity $v = 0$, and v increases with distance from the wall but at a decreasing rate; at a certain distance from the wall, the fluid velocity will reach a maximum. Considering phases 1 and 2 in figure 6.7, which are at a distance Δx apart, the velocities along the phases will be v_1 and v_2; if $v_2 > v_1$ then $\Delta v = v_2 - v_1$. Hence, the velocity gradient

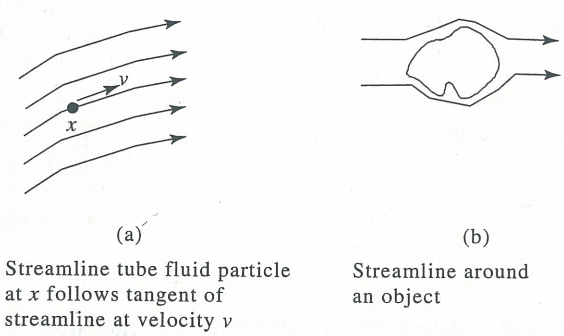


Figure 6.6 Illustration of streamlines.

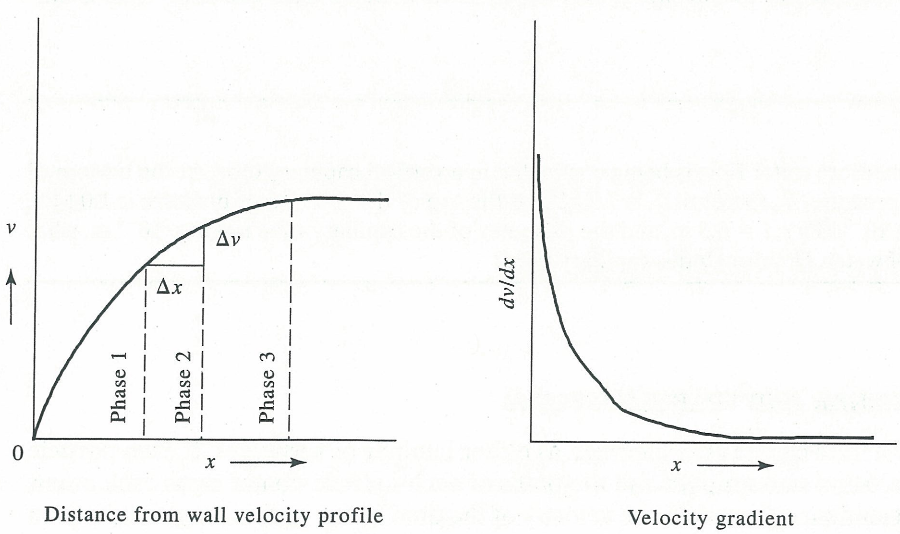


Figure 6.7 Illustration of velocity profile (left) and velocity gradient (right); v is velocity, x is distance, and dv/dx is velocity gradient, that is, change in velocity with change in distance.

dv/dx may be expressed as

$$\frac{dv}{dx} = \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \tag{6.15}$$

The velocity gradient as illustrated in figure 6.7 is the reciprocal of the slope of the velocity profile. Since x is the measured distance perpendicular to the direction of flow, and from the definition of velocity we may state that

$$\frac{dv}{dx} = \frac{d\left(\frac{dy}{dt}\right)}{dx} = \frac{d\left(\frac{dy}{dx}\right)}{dt} \tag{6.16}$$

where dy/dx is the shear at phase 2. Foregoing significant detail, the velocity gradient is the time rate of shear. As a consequence, when $dy = 0$, the shear vanishes and the velocity gradient also vanishes. Because real fluids resist shear, shear forces must always exist whenever there is a time-rate of shear. For more in-depth knowledge of shear and shear stress, the reader is referred to the literature discussing the science of rheology and related behavior (see the suggested readings section at the end of the text).

Osborne Reynolds was the first to demonstrate the difference between laminar flow and turbulent flow in 1883. Named in his honor, the Reynolds number, R_e, expresses the ratio of inertial forces to viscous forces during flow and is widely used to differentiate between laminar flow (low velocities) and turbulent flow (high velocities). For example, for a Reynolds number of value less than about 5, a condition of linear laminar flow exists; for values of about 5 to 100 the flow is termed nonlinear laminar, and for values greater than 100, flow is turbulent (Bear 1972). These general classifications assume that flow is occurring through a medium, and not a pipe or open channel. For Reynolds numbers less than about 5, viscous forces dominate; for values greater than 5 (up to about 100), inertial forces dominate. At values greater than 100, laminar flow is assumed to give way to turbulent flow. The Reynold's number is often used to determine the magnitude of friction loss within a system.

For flow through soil, the Reynolds number is expressed as

$$R_e = \frac{\rho q l}{\eta} \tag{6.17}$$

where ρ is the fluid density (kg m^{-3}), q is the flux density (some hydrologists refer to this as the specific discharge; m s^{-1}), l is a representative length-dimension of the medium in question (usually taken as the pore diameter or mean-particle diameter), and η is the viscosity (kg m^{-1} s^{-1}; converted from Pa s^{-1} listed in Table 6.1 for convenience of calculation). If R_e remains constant, fluid flow will be steady. For a detailed discussion of the Reynolds number and its application to flow through soils, the reader is referred to Bear (1972).

QUESTION 6.7

Ten cubic cm of water at 25 °C is passed through a steel capillary tube 30 cm length and 1.5 mm diameter in 4 seconds. What is the pressure required, and the Reynold's number?

SUMMARY

In this chapter we discussed the basic principles of water flow in soils, and how Bernoulli's expression relates pressure to fluid velocity, elevation, and conservation of energy when applied to a perfect fluid. In addition, the state of a fluid at any point may be characterized by four quantities: **(1)** pressure, P; **(2)** velocity, v; **(3)** elevation, h; and **(4)** density, ρ. Also described was Torricelli's law, which states that the speed of efflux for an open system is equal to the speed gained by a free-falling body through a vertical distance h. Additionally, Poiseuille's law, describing the laminar flow of a fluid in a small cylindrical tube, was discussed. Poiseuille's law states that the discharge rate, Q, of a fluid in a cylindrical tube of small, fixed radius, R, is dependent upon the driving force acting on the fluid as well as upon the internal friction forces between molecules within the fluid, characterized by the fluid viscosity, η. Finally, it was discussed how the movement of a fluid can be characterized as either laminar or turbulent. If a particle of fluid flows along a smooth path and the path of this particle and others do not cross, the flow is termed laminar, and the velocity of the fluid at any point along its flow-path remains constant in time. Turbulent flow was defined as an irregular flow, often characterized by small whirlpool-like regions.

ANSWERS TO QUESTIONS

6.1. Assume A_1 is the cross-sectional area of the hole and v_1 is the velocity of fluid exiting the hole, then $A_2 >> A_1$ and $v_2 << v_1$. Also assume $v_2 \approx 0$ and $P_1 = P_2 = P_a$. Thus,

(a)

$$P_1 + \frac{\rho v_1^2}{2} + \rho g y_1 = P_2 + \frac{\rho v_2^2}{2} + \rho g y_2$$

$v_1 = [2g(y_2 - y_1)]^{1/2} = [2(9.80)(20)]^{1/2} = 19.8$ m/s.

(b) The flow rate is $A_1 v_1 = (\pi d^2/4)(19.8) = 4.2 \times 10^{-5}$ m^3/s. Solve for d to obtain 1.64×10^{-3} m or 1.64 mm.

6.2. By utilizing Bernoulli's equation, the pressure is converted entirely to kinetic energy, which is converted into gravitational potential energy. Thus,

$$\Delta P \rightarrow \frac{1}{2}\rho v^2 \rightarrow \rho g h$$

where $\rho = 1000$ kg/m^3. Hence, **(a)** $\rho g y = (10^3)(9.80)(40 \text{ m}) = 1/2\ \rho v^2$; solving for v, we obtain $v = 28$ m/s. **(b)** $\Delta P = (10^3)(9.80)(40 \text{ m}) = 3.92 \times 10^5$ Pa or 3.87 atm.

6.3. To obtain the solution we may use the formula $P = P_a + \rho g h$. Thus, $P = (1.01 \times 10^5 \text{ Pa}) + (1.0 \times 10^3 \text{ kg/m}^3)(9.80 \text{ m/s}^2)(500 \text{ m}) = 5.0 \times 10^6$ Pa. This is roughly 50 times greater than atmospheric pressure.

6.4. For the solution to this problem, the equation used in question 6.1 becomes $P = \rho g h = \rho g (H - y)$, and to find the force exerted by the fluid over a specific surface area ΔA we may use

$$P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}$$

Consequently, the force is given by $dF = P\,dA = \rho g(H - y)w\,dy$ and the total force on the dam is

$$F = \int P\,dA = \int_0^H \rho g(H - y)w\,dy = \frac{1}{2}\rho g w H^2$$

Hint: draw a diagram of the dam's surface, shade to the height of a chosen water level, then select a small strip across the width of the shaded face to calculate pressure. Remember, the total force on the dam must be obtained from the expression $F = \int P dA$, where dA is the area of the small strip. As a result: w is the dam width; H is the total height of water; h is the height of water above the small strip; y is the depth of water below the small strip; and, of course, dy is the thickness of the small strip.

6.5. This is a classic case of Torricelli's law. First, assume the tank is large in cross-sectional area compared to the exit hole, that is, $A_2 >> A_1$. Thus, the fluid will be relatively at rest at the top of the tank (point 2). Now, applying Bernoulli's equation to points 1 and 2 and at the exit hole $P = P_a$, we obtain

$$P_a + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P + \rho g y_2$$

however, since $y_2 - y_1 = h$, this will reduce to

$$v_1 = \sqrt{\frac{2(P - P_a)}{\rho} + 2gh}$$

Since A_1 is the area of the exit hole, the flow rate from this hole is $A_1 v_1$. When the pressure P is large compared to atmospheric pressure, the term $2gh$ can be neglected and the speed of efflux is primarily a function of P. Also, if the tank is open to atmospheric pressure, $P = P_a$ and $v_1 = (2gh)^{1/2}$.

6.6. Using Poiseulle's law (equation 6.14), we find that $Q = [\pi(1.8 \times 10^{-4}\ \text{m})^4/(8)(10^{-6}\ k\text{Pa} \cdot \text{s})] \times [(7.0\ k\text{Pa} - 1.0\ k\text{Pa})/0.5\ \text{m}] = 4.947 \times 10^{-9}\ \text{m}^3/\text{s}$, or 0.297 mL/min.

6.7. The pressure, P, required to force the fluid through the capillary tube is

$$P = (8Vl\eta)/(\pi r^4 t)$$
$$= [(8)(10 \times 10^{-6}\ \text{m}^3)(0.30\ \text{m})(8.90 \times 10^{-4}\ \text{kg m}^{-1}\ \text{s}^{-1})]/[\pi(7.5 \times 10^{-4}\ \text{m})^4\ (4\ \text{s})]$$
$$= 5373\ \text{N m}^{-2}.$$

To calculate the Reynolds number, $R_e = \rho q l/\eta$, we must first obtain q, thus

$$q = [(10 \times 10^{-6}\ \text{m}^3)/(4\ \text{s})]/\pi(7.5 \times 10^{-4}\ \text{m})^2 = 1.4147\ \text{m s}^{-1}.$$
$$R_e = (1.5 \times 10^{-3}\ \text{m})(1.4147\ \text{m s}^{-1})(0.99708 \times 10^3\ \text{kg m}^{-3})/(8.90 \times 10^{-4}\ \text{kg m}^{-1}\ \text{s}^{-1})$$
$$= 2377.1.$$

A value of this magnitude would indicate turbulent flow.

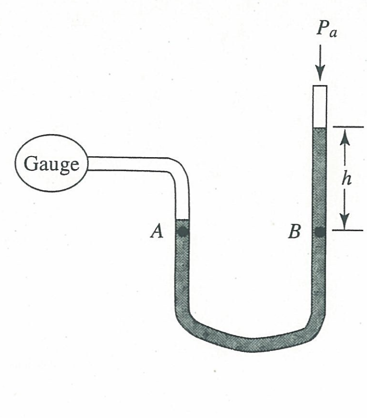
ADDITIONAL QUESTIONS

6.8. You have a field-study site near the western coast of the United States. An oceanographer asks your assistance in calculating the pressure 1000 m beneath the ocean's surface. Assume water density is 1.0×10^3 Kg/m^3 and $P_a = 1.01 \times 10^5$ Pa.

6.9. At what depth in a lake is the absolute pressure three times the atmospheric pressure?

6.10. In Greenland, the ice sheet is 1 km thick. What is the pressure on the ground beneath the ice? Assume $\rho_{ice} = 920\ \text{kg/m}^3$.

6.11. You are calibrating pressure transducers in the laboratory with a u-shaped tube (see figure below). What is the absolute pressure, P, on the left side if $h = 20$ cm? What is the gauge pressure?



6.12. The rate of flow through a horizontal pipe is 1.5 m^3/min. What is the velocity of flow at a point where the pipe is **(a)** 5 cm and **(b)** 2 cm?

6.13. Water flows through a 6.35-cm diameter hose at a rate of 0.012 m^2/s. At what velocity does water exit the nozzle at the end of the hose?

6.14. A Venturi tube can be used as a fluid flow meter. If $P_1 - P_2 = 21 \times 10^3$ Pa ($\approx$ 3 lb/in^2), what is the flow rate (m^3/s) if the outlet radius is 1 cm, and inlet radius of the tube is 2 cm? Assume fluid density is 700 kg/m^3.

6.15. You are working with an above-ground storage tank (AST) which is filled to a height h_o. If this tank is punctured at a height h from the bottom of the tank, how far from the tank will the stream land? Assume $h_o = 5$ m and $h = 2$ m.

6.16. What is the Reynolds number for flow of a liquid in a 1.2-cm diameter soil pore? The fluid is from an oil spill: $\rho = 850$ kg/m^3; viscosity is 0.3 Pa · s; and velocity is 3.0×10^{-5} m/s.

6.17. You determine a fluid viscosity at 40 °C by measuring flow rate through a capillary tube at a known pressure difference between the ends. The capillary radius is 0.70 mm, length is 1.5 m. When a pressure difference of 1/20 atm is applied, a volume of 292 cm^3 was collected in 10 minutes. What is the viscosity of the fluid? Identify the fluid.