Unsaturated Zone Hydrology for Scientists and Engineers

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8

Unsaturated Water Flow in Soil

INTRODUCTION

So far in this text, we have discussed the physical properties and characteristics of soil, and why these properties are important in chemical transport. We also discussed the behavior of clay–water systems as well as the importance of electrokinetic properties, ion exchange, swelling, dispersion and other parameters associated with clay behavior. We then discussed the aspects of potential and thermodynamics of soil water, including capillary theory, chemical and matric potential, and hysteresis. A brief review of the chemical properties associated with soil water and the basic principles involved in water flow followed, then a very brief review of saturated water flow, including various flow cases that adhere to (or can be explained by) Darcy's Law. Although water flow in saturated soils and other media is important, soils are usually not saturated with fluid in typical vadose-zone investigations. As a consequence, the principles previously discussed are necessary in order to understand the concepts involved in water flow and the transport of chemicals in the unsaturated zone.

In chapter 7, we discussed how saturated flow depends on a positive hydraulic gradient, that is, a combination of the gravitational (elevation) and pressure potentials of the fluid that will cause water flow from high potential to low potential. Within the unsaturated zone the pore spaces are not completely filled with fluid, thus, the effective conducting pore space is much smaller than if the medium were saturated and the pore space was normally filled with both liquid and gas phases. Water flow under these conditions is termed unsaturated flow.

The unsaturated zone itself can be either very shallow or very deep, depending on geographic location. In the eastern United States, it is typical to find shallow water tables in which the root-zone portion of the unsaturated zone may be of major concern, especially since the presence of plants and greater populations of bacteria and other microbes make this shallow zone (usually about 1 m) much more dynamic. In contrast, the vadose zone in the western United States is deeper, anywhere from many meters to hundreds of meters thick. A schematic of the unsaturated zone is shown in figure 8.1. Regardless of depth the principles involved are the same, although deeper vadose zones generally require more expense in instrumentation and data collection.

Because soil pores are generally filled with both liquid and gas within the unsaturated zone, the degree of saturation refers to the portion of pore volume filled with water. Also, since the largest water-filled pores empty out first, the unsaturated hydraulic conductivity decreases rapidly as the volumetric water content decreases. This is due to the fluid that is
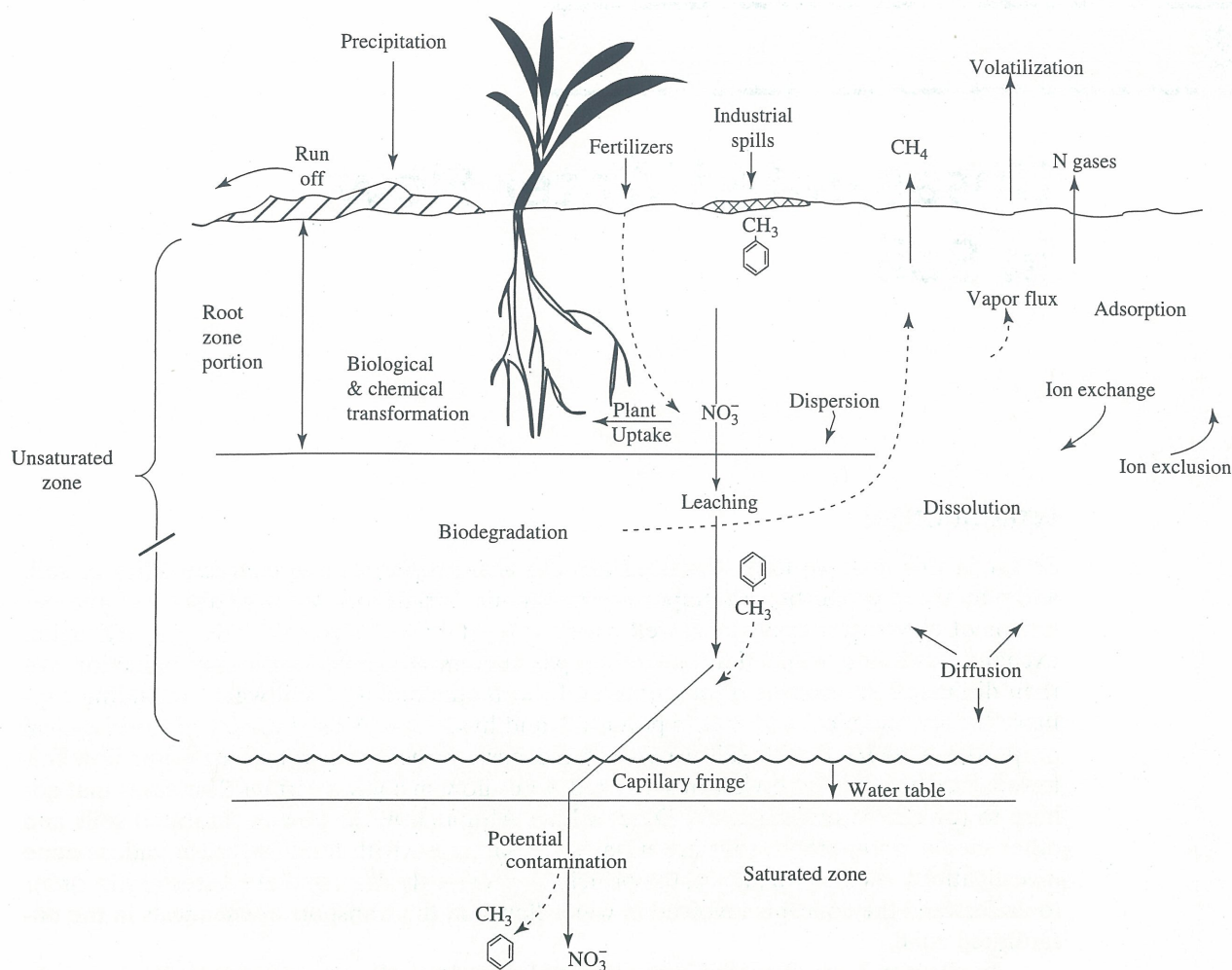
Figure 8.1 Simplified schematic of unsaturated zone processes

constrained to smaller flow channels as water content decreases. The channels not only become narrower, but the flow path becomes more tortuous and fluid can literally flow as a hydration film from one position to the next. Fluid flow in the unsaturated zone primarily is due to water content, matric potential gradient (also known as soil pressure or capillary potential), and gravitational potential. As opposed to a positive head (as discussed in saturated fluid flow), flow in the unsaturated zone is usually by a negative gradient, as well as the other parameters mentioned above. (See chapter 4 for a review of potential.)

In this chapter we discuss the validity of Darcy's Law for unsaturated conditions; factors affecting unsaturated conductivity; basic flow equations for unsaturated flow; and some fundamental mathematics that we use to solve the general unsaturated flow equation. The general continuity equation (as derived in chapter 10, question 10.2) must be altered to describe unsaturated flow fully; this alteration results in a nonlinear differential equation. A nonlinear differential equation is one in which the sum of the independent solutions, each multiplied by an arbitrary constant, does not bring about a solution to the equation. Due to this nonlinearity, the mathematics of unsaturated flow are highly complex. An additional major concern is that Darcy's Law may not apply for all unsaturated flow conditions.

8.1   VALIDITY OF DARCY'S LAW FOR UNSATURATED CONDITIONS

As discussed in chapter 7, we express Darcy's Law as

$$q = -K(\theta)i \tag{8.1}$$

where q is the flux density (volume of water flowing across a unit area per unit time), $K(\theta)$ is the conductivity (and in this case is termed the capillary conductivity to differentiate it from saturated hydraulic conductivity), and i is the hydraulic gradient (typically expressed as $\Delta H/\Delta L$). Equation 8.1 implies that if $K(\theta) \neq 0$ and a hydraulic gradient exists, flow will occur at a rate proportional to $K(\theta)$, assuming i is constant. If a gradient (no matter the size) is applied to fluid in an unsaturated medium and the water does not move, or the velocity (v) is not a straight-line function of the hydraulic gradient, or K does not vary as θ varies, then Darcy's Law will not apply and the fluid is termed non-Newtonian in behavior. However, fluids *not* conforming to Darcy's law are not necessarily non-Newtonian (Bear 1972). Often, the fluid simply does not flow until the hydraulic gradient reaches a certain value. In this case, the value the gradient reaches when flow begins is called the threshold gradient.

We do not understand fully the extent to which non-Darcy flow exists in soil, but we do know that many forces affect the flow of water within a medium. Among these is the hydrogen and covalent bonding between the water and clay particles, that in some instances is sufficient to prevent flow (in the presence of a very small hydraulic gradient). Additional forces can include: ion adsorption; double-layer thickness; and streaming potential (all discussed in chapter three). Soil structure factors such as the presence of macropores (cracks, worm holes, etc.); preferential flowpaths (areas with lower bulk density than the surrounding soil matrix); fingering; very structured and aggregated media; sand lenses; and other heterogeneities, may also lead to non-Darcy flow. However, understanding is still limited on the effects of these conditions on unsaturated flow and transport, although under investigation for more than a decade.

The continuity equation, while commonly used to simulate unsaturated flow and transport, does not consider the validity of Darcian flow. As a result, the continuity equation (in three dimensions)

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) = \frac{\partial \theta}{\partial t} \tag{8.2}$$

can be used without having to consider the limitations placed on Darcy's Law. However, when non-Darcy flow occurs (whether due to fluid property or structural effects), models based on the continuity equation are quite poor in their prediction of water and solute transport. When this and other complex relations (between matric potential, volumetric water content, and unsaturated hydraulic conductivity) are taken into account, vadose-zone experimentation frequently involves extensive instrumentation, rigorous mathematics, and methods of analysis that typically include numerical approximations for solution of a problem.

8.2   FACTORS AFFECTING UNSATURATED HYDRAULIC CONDUCTIVITY

We discussed several flow cases for saturated-flow problems in chapter 7. To show factors affecting unsaturated water flow, we begin this chapter discussing an unsaturated flow system where the pores are filled with a mixture of air and water (i.e., the pores are only partially water filled) and water flows under the influence of a negative suction or matric potential. A schematic of the flow system is shown in figure 8.2. For this case, the hydraulic head H at the inflow (point A) is $-h_1 + z_1$, where $-h$ is the matric potential and z is the gravitational
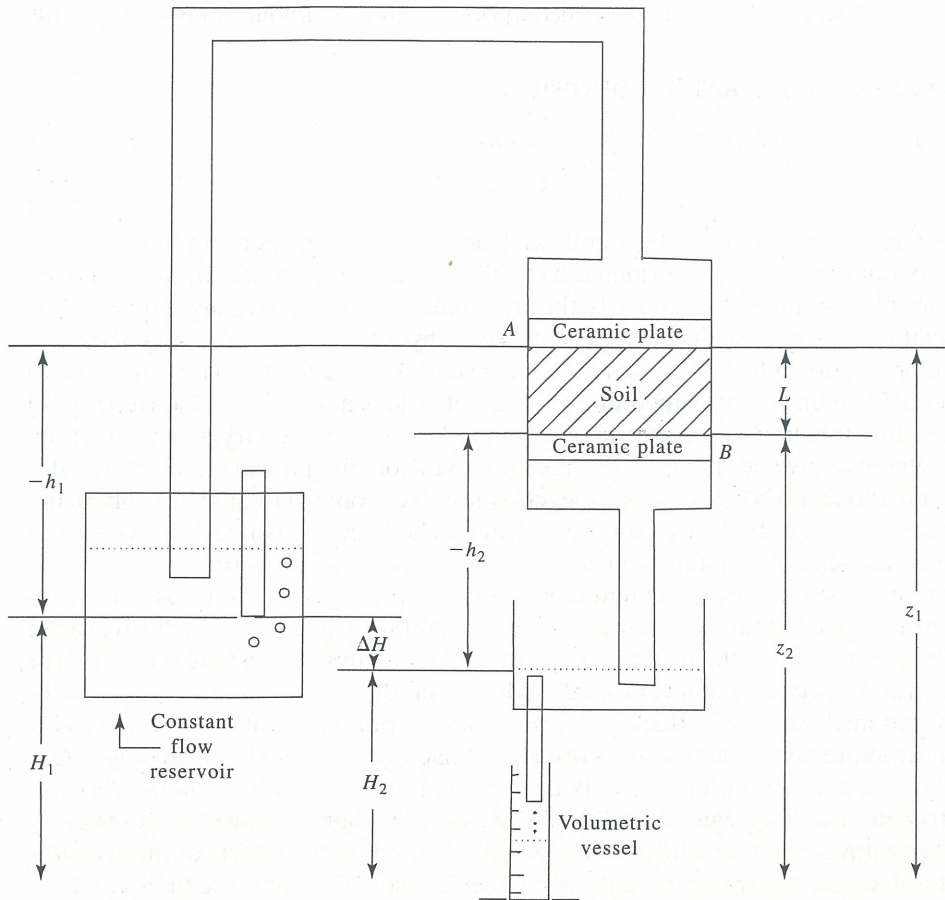
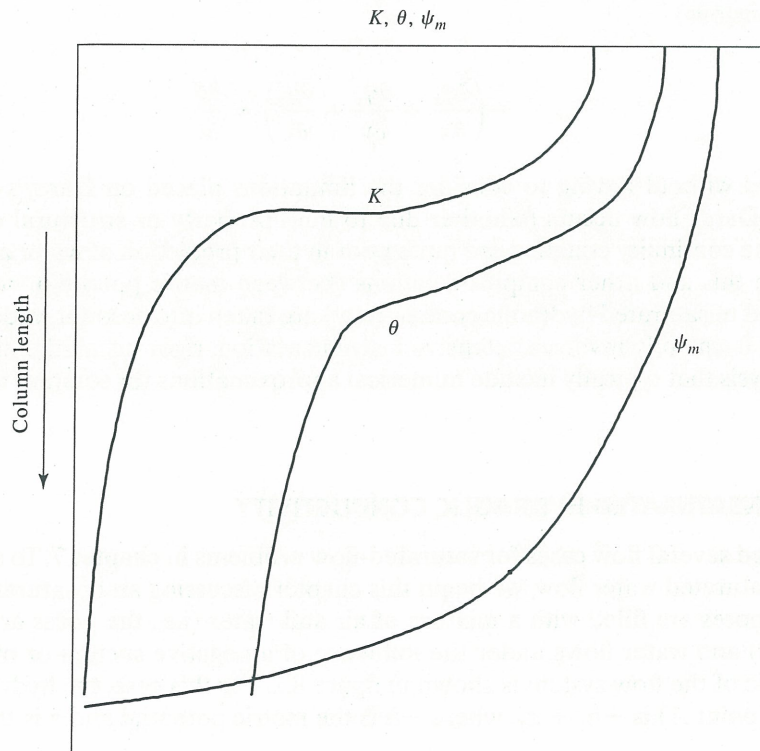Figure 8.2 Illustration of flow system under the influence of matric potential (i.e., negative suction)

In the figure, labels include:

A, Ceramic plate, Soil, Ceramic plate, B, L, $-h_1$, $-h_2$, ΔH, z_1, z_2, H_1, H_2, Constant flow reservoir, Volumetric vessel

K, θ, ψ_m

Column length

K

θ

ψ_m

Figure 8.3 Illustration demonstrating non-uniform wetness and thus, variation in unsaturated hydraulic conductivity, K, volumetric water content, θ, and matric potential, ψ_m (after Hillel 1971)

potential; the hydraulic head H at the outflow (point B) is $-h_2 + z_2$. Subtraction of the two heads will give a change of hydraulic head: $\Delta H = H_1 - H_2 = -(h_1 - h_2) + (z_1 - z_2)$.

Assuming a constant matric potential along the column, we surmise that flow is steady, since the column is short and the flux through the column follows Darcy's law (equation 8.1). However, for an unsaturated medium (even in a short column but especially for longer soil columns), it is unlikely that the gradient is constant along the column length. As a rule, this is a result of non-uniform wetness along the column. Consequently, the volumetric water content, matric potential, and unsaturated hydraulic conductivity all vary with distance in the column. Figure 8.3 represents this graphically, and also illustrates that as matric potential decreases (increased suction), the suction gradient usually increases with a subsequent decrease in capillary conductivity along the column. Since the hydraulic gradient along the column length is not constant, we do not obtain the capillary conductivity by dividing the flux by the gradient ($\Delta H / \Delta L$), as we do in a saturated system. Instead, the flux has to be divided by the specific gradient at a given location; thus, we have to obtain a flow solution iteratively.

We have discussed a soil column that is a single layer, basically of one soil type up to this point. We now consider what happens to matric potential and moisture content in a stratified column, which is more applicable to field situations in which there is a layered profile. Using a layered column with a coarse media such as sand for layer one and a fine medium like clay for layer two (in contact with a water table), and assuming a steady downward flow, we find a distinct discontinuity in moisture content at the interface of the two layers, although the flux rate is constant throughout the column (see figure 8.4a). The flux rate is accommodated by a
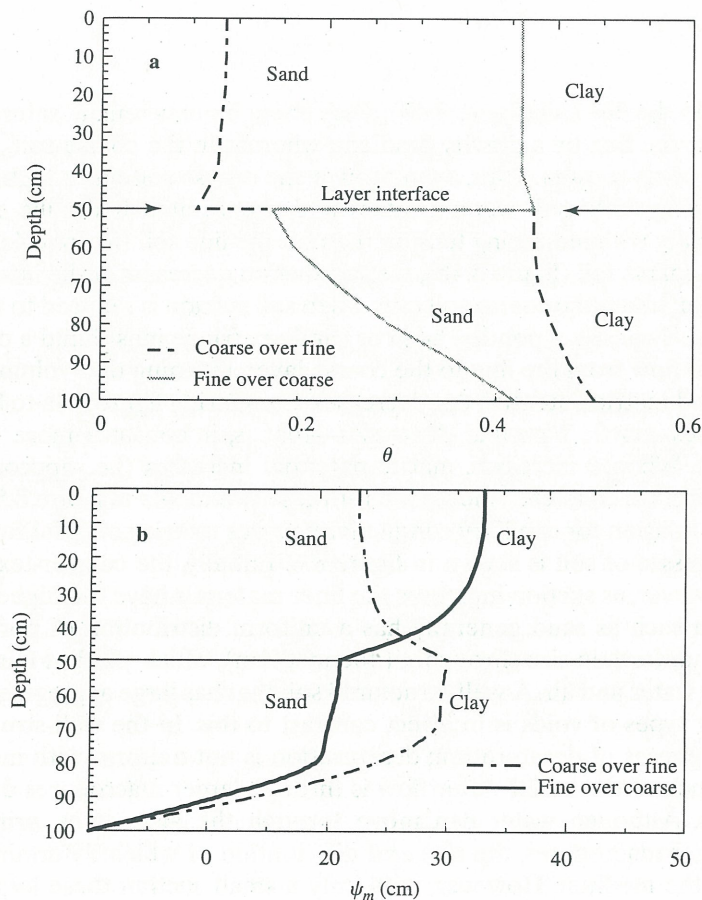

Figure 8.4   Discontinuity of θ at soil interface in a layered column with two different soil textures (a), and relation of θ to ψ_m using same soils (b).
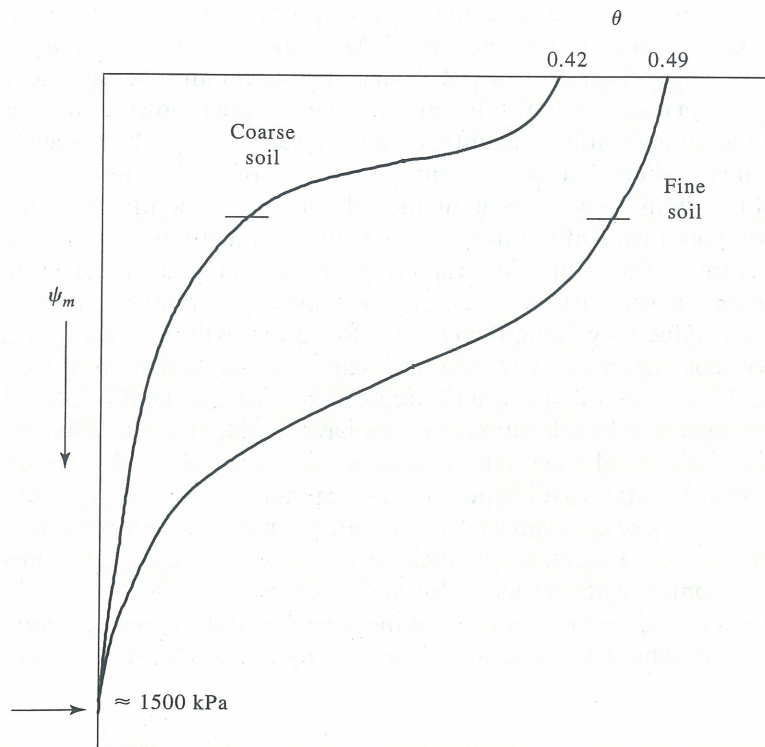
Figure 8.5   Soil moisture curves for soils shown in figure 8.4

greater tension in the fine soil (figure 8.4b), since enough pores remain saturated at that tension to transport the flux by a gravity gradient; whereas in the coarse soil, fewer pores are available at the same tension. Thus, even though the coarse soil has a higher saturated hydraulic conductivity, both soil water content and unsaturated hydraulic conductivity decrease more rapidly with increasing tension than for the fine soil (figure 8.4a). When the fine soil overlies the coarse soil (figure 8.4b), the soil suction decreases at the interface of the two layers, since water enters the coarse soil only when soil suction is reduced to that of the pores in the coarse soil. Typically, a ponded head or capillary fringe must build a positive pressure to allow water to flow from the fine to the coarse layer, meaning that volumetric water content increases and matric potential, ψ_m, decreases. Comparing figure 8.4a to 8.4b, we see that when θ decreases, matric potential decreases—that is, it becomes more negative (has a greater suction). When θ increases, matric potential increases (i.e., approaches zero). We show the soil moisture characteristic curves for these two media in figure 8.5.

A general relation for capillary conductivity versus matric potential for three different (non-layered) classes of soil is shown in figure 8.6. Initially, the coarser-textured soil has a higher $K(\theta)$; however, as suction increases the finer materials have the higher $K(\theta)$.

A medium such as sand generally has a uniform distribution of pore sizes (i.e., the pores are fairly uniform in size throughout the medium), which yields a fairly uniform mixture of soil-pore water and air. A well-structured soil that has large aggregates, cracks, macropores, and other types of voids is in direct contrast to this. In the well-structured medium, there are more phases of desaturation; desaturation is not uniform with increased suction. For example, much of the initial water flow is through larger macropores during significant recharge events. Although water can move through the aggregates, primarily it moves through the larger macropores, the size and distribution of which determines the capillary conductivity of the medium. However, with only a small suction these larger pores empty
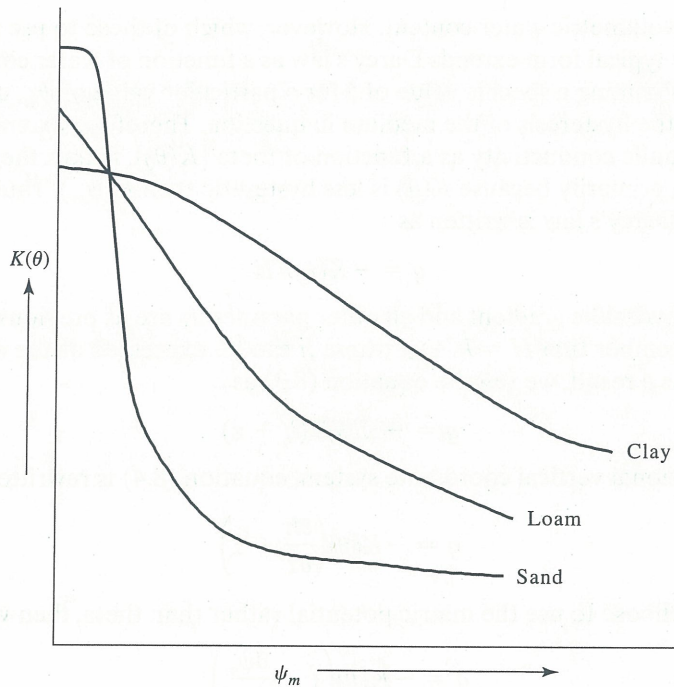
Figure 8.6   General relation for capillary conductivity ($K(\theta)$) versus matric potential (ψ_m) for three different soil classes

very rapidly, leaving interped and inter-aggregate pores still saturated. It is at this phase that the water within the medium begins to move as hydrated-water films since the area of conductivity (i.e., the number of pores conducting water) rapidly decreases. Thus, even though the aggregates are saturated, the overall conductivity of the entire system is very low. As we discussed earlier, it is unlikely that full saturation is achieved due to air entrapment (likely to occur during significant recharge events). This discussion shows that Darcy's law is non-linear under unsaturated conditions.

Unsaturated hydraulic conductivity is affected by the presence of swelling clay soil. As water content decreases with increasing suction, the soil shrinks, reducing pore size and then K. Good examples of this are well-structured vertisols, that have a high K during a state of low-moisture content but due to swelling after a significant recharge event, are reduced dramatically, to almost zero. Therefore, in general terms, the factors that can cause distinct reductions in unsaturated conductivity with an increase in soil suction are: reduction in pore conductivity due to water loss from the largest pores; reduction in effective porosity; increased tortuosity; as well as the fact that water near clay surfaces can have a viscosity four times higher than in bulk solution (see chapter 3). Temperature effects, more pronounced at water contents from about 0 to -30 kPa, result in higher K at higher temperatures, due to decreased viscosity and surface tension. Because of clay–water interactions as discussed briefly above (and in chapter 3), the effect of temperature is less dominant at low-water content.

8.3   DEVELOPMENT OF UNSATURATED FLOW EQUATIONS

The conventional form for Darcy's law does not describe adequately water flow in the vadose zone, due to the rapid decrease of both hydraulic conductivity with decreasing water content and total cross-sectional area available for water flow in an unsaturated medium. Consequently, it has been extended; this extension assumes that K is a function of matric

potential and/or volumetric water content. However, which of these to use is still somewhat controversial; the typical form extends Darcy's law as a function of water content. This is due to the difficulty obtaining a specific value of θ for a particular value of ψ_m, unless we know a great deal about the hysteresis of the medium in question. Therefore, extending Darcy's law and writing hydraulic conductivity as a function of theta $[K(\theta)]$, makes the problem of hysteresis avoidable, primarily because $K(\theta)$ is less hysteretic than $K(\psi_m)$. Thus, in tensor form, the extension of Darcy's law is written as

$$q = -K(\theta)\nabla H \tag{8.3}$$

where ∇H is the hydraulic gradient and all other parameters are as previously discussed. It is important to remember that $H = h + z$, where h can be expressed as the suction or matric potential (ψ_m). As a result, we rewrite equation (8.3) as

$$q = -K(\theta)\nabla(h + z) \tag{8.4}$$

For a one-dimensional vertical coordinate system, equation (8.4) is rewritten as

$$q = -K(\theta)\left(\frac{\partial h}{\partial z} + 1\right) \tag{8.5}$$

Also, should we choose to use the matric potential rather than theta, then we obtain

$$q = -K(\theta)\left(1 - \frac{\partial \psi_m}{\partial z}\right) \tag{8.6}$$

Equation 8.6, assuming ψ is a single-valued function of θ, $\partial \psi_m / \partial z$ is rewritten as follows, using the chain rule of calculus:

$$\frac{\partial \psi_m}{\partial z} = \frac{d\psi_m}{d\theta}\frac{\partial \theta}{\partial z} \tag{8.7}$$

The first term on the right-hand side of equation 8.7 is the inverse of the specific water capacity, that is, the reciprocal of change in water content per unit change in soil suction or matric potential. The second term is the water-content gradient, with respect to depth.

By substituting equation 8.7 into equation 8.6, we have

$$q = -K(\theta)\left(1 - \frac{\partial \psi_m}{\partial \theta}\frac{\partial \theta}{\partial z}\right) \tag{8.8}$$

which is written as

$$q = -K(\theta) - K(\theta)\left(-\frac{\partial \psi_m}{\partial \theta}\right)\frac{\partial \theta}{\partial z} \tag{8.9}$$

Equation 8.9 is Richards' equation (Richards 1931), and is written so that the hydraulic diffusivity term is readily introduced into the basic flow equation, as described below.

According to Poiseuille's law (discussed in chapter 6), flow is related to pore radius. Consequently, if the conducting pore size is reduced by one-half, the capillary conductivity will be reduced by one-quarter. As more water is removed so that transport is by hydrated films, the effective path length over which the fluid travels is lengthened, that is, the fluid cannot go straight through the medium but must meander around individual particles that are covered with the hydrated-water films. This results in a tortuous path of flow that decreases conductivity further. As continuity fails within the pore system, no fluid flow occurs. At this point, only vapor transport occurs within the system; however, vapor flow is usually minimal unless there are significant temperature gradients.

By considering a bundle of capillary tubes similar to those of Poiseuille's law, but taking into account the tortuosity of the medium, we rewrite a generalized form of Kozeny's equation to determine K, such that

$$K = \frac{\rho_w g}{k' \eta s^2} \left(\frac{L}{L_e}\right)^2 \left[\frac{\phi_e^3}{(1 - \phi_e)^2}\right] \tag{8.10}$$

where ρ_w is density of water (g cm^{-3}), g is the gravitational constant, k' is the pore-shape factor (ranging from 2 to 2.5), η is the viscosity of water $(\text{Pa} \cdot \text{s})$, s is the specific-surface area (ratio of total surface area of solid to volume of same solid; $\text{cm}^2 \text{cm}^{-3}$), ϕ_e is the effective porosity (area of conducting pore or channel per unit area of cross-section—sometimes expressed as $\theta - \phi_r$; where ϕ_r is the residual moisture content), L is length of column or profile (L), and L_e is the effective length (L). The ratio of length to effective length may be determined by

$$\left(\frac{L}{L_e}\right)^2 = \frac{E_o}{\phi_e E_s} \tag{8.11}$$

where E_s is a measurement of the electrical resistivity of the soil $(\text{S m}^{-1}$—Siemens per meter) when the pore space is filled with a liquid of known electrical resistivity, E_o. Using equation 8.10, we can account for both the pore-shape factor and tortuosity, when determining K. The generalized Kozeny's equation 8.10 is a semiempirical equation relating K to θ and to ψ and is based on the assumption that tortuosity increases as a power of $1/S_e$, where $S_e = \theta - \theta_r/\phi - \theta_r$; see equation 8.12f.

Since the hydraulic conductivity depends on the volumetric water content and/or matric potential, we have modified Darcy's law for unsaturated media, with the more general forms as expressed in equations 8.3 and 8.4 (where K is written as a function of θ). Since no universal relations are available for capillary conductivity versus soil suction or water content, several empirical relations are proposed; these relations follow.

$$K(\psi_m) = \frac{a}{\psi_m} \qquad \text{(Baver, Gardner, and Gardner 1972)} \tag{8.12a}$$

$$K(\psi_m) = a(b + \psi_m^n)^{-1} \qquad \text{(Childs and Collis-George 1950a)} \tag{8.12b}$$

$$K(\psi_m) = \frac{K_s}{\left[1 + \left(\dfrac{\psi_m}{\psi_c}\right)^n\right]} \qquad \text{(Gardner 1958)} \tag{8.12c}$$

$$K(\psi_m) = \frac{K_s}{b + \psi_m^n} \qquad \text{(Childs and Collis-George 1950b)} \tag{8.12d}$$

$$K(\theta) = a(\theta)^n \qquad \text{(Marshall and Holmes 1979)} \tag{8.12e}$$

$$K(\theta) = K_s \left(\frac{\theta - \theta_r}{\phi - \theta_r}\right)^n \qquad \text{(Brooks and Corey 1966)} \tag{8.12f}$$

$$K(\theta) = K_s \exp(a\psi_m) \qquad \text{(Mualem 1976)} \tag{8.12g}$$

$$K(\theta) = K_s \sqrt{\frac{\theta - \theta_r}{\phi - \theta_r}} \left[1 - \left(1 - \left(\frac{\theta - \theta_r}{\phi - \theta_r}\right)^{1/m}\right)^m\right]^2 \qquad \text{(Van Genuchten 1980)} \tag{8.12h}$$

where $m = 1 - 1/n$; $K(\theta)$ is the unsaturated hydraulic conductivity; K_s is the saturated hydraulic conductivity for the same medium; $a, b, m,$ and n are empirical constants (for fine textured media $n = 1$–2 and can be up to 4 or more for coarse media); ψ_c is the matric

potential for which $K = 1/2(K_s)$; and θ_r is the residual saturation (where the moisture characteristic curve goes vertical; i.e., the water content will not get much lower). Note that equation 8.12a is not used for soil near saturation, that is, where the matric potential is near zero. Also, the relation of conductivity to matric potential depends on hysteresis, which has to be considered in the rigorous analysis of unsaturated-flow problems. For further information about equations 8.12a to 8.12h, we refer the reader to the respective references for each equation. Also, equations 8.12a to 8.12h are of different categories based on the information required to determine their coefficients (those requiring empirically developed values of K versus θ or ψ) and those based on mathematical analysis of the moisture-characteristic curve.

QUESTION 8.1

You conducted an experiment using some large sandy soil cores (97% sand) in the laboratory. The cores were set up with a matric suction (−20 kPa) applied to the bottom, across a porous ceramic plate. You assume that the suction applied was equal throughout the core. From soil samples taken at the site (where the cores were extracted), you run standard texture, soil moisture, organic carbon tests, et cetera. Using data from the soil-moisture characteristic curve in a computer program [after principles developed by Millington and Quirk (1961)] to obtain an unsaturated hydraulic conductivity curve, you determine the following: $\psi_c = 0.55$ m, $K_s = 16.72$ m/day, and the empirical constant for this soil, $n = 4.0$. What is $K(\psi)$?

8.4   HYDRAULIC DIFFUSIVITY

Working from the previous discussion, we now introduce the hydraulic diffusivity, D, often referred to as the soil-water diffusivity. Because both K and ψ_m are assumed to be single-valued functions of θ, D is expressed as

$$D(\theta) = K(\theta)\left(\frac{d\psi_m}{d\theta}\right) \tag{8.13}$$

where D is expressed in units $L^2\,T^{-1}$. Equation 8.13 is rewritten as

$$D(\theta) = \frac{K(\theta)}{\left(\dfrac{d\theta}{d\psi_m}\right)} \tag{8.14}$$

From equation 8.14 we see that D is the ratio of the unsaturated hydraulic conductivity to the specific moisture capacity (the denominator of equation 8.14), that is, a change in volumetric water content per unit change in matric potential. The denominator sometimes is expressed mathematically in "shorthand" as $c(\theta)$. The specific moisture capacity (hydraulic diffusivity) is also considered the reciprocal of the slope of the moisture characteristic curve at the same water content.

Substitution of equation 8.13 into equation 8.9 gives

$$q = D(\theta)\frac{\partial \theta}{\partial z} - K(\theta) \tag{8.15}$$

However, if we were to express D for horizontal flow in which the effects of gravity can be neglected, then equation 8.15 is rewritten as

$$q = -D(\theta)\frac{\partial \theta}{\partial x} \tag{8.16}$$

which expresses a measure of moisture flow due to a moisture or specific water-content gradient. Equation 8.16 is analogous to Fick's law of diffusion or Fourier's law of heat flow. However, we have to remember that equation 8.16 is written assuming that K, ψ_m, and $(d\psi_m/d\theta)$

are unique or single-valued functions of θ—not exactly true for unsaturated flow because of hysteresis in each of these parameters (i.e., the wetting and drying history of the medium; entrapped air; overburden pressure; water-contact angle, and so on). Essentially, the value of D for a drying medium is likely different than that for the same medium upon wetting.

The introduction of $D(\theta)$ allows us more readily to solve unsaturated flow problems because the flow equations are of a form similar to the diffusion equation, for which analytical solutions are available. Also, we can readily substitute equation 8.16 in the solute transport equation (see chapter 10). By introducing D into the differential flow equation (again, see chapter 10) and assuming only horizontal flow, we have

$$\frac{\partial \theta}{\partial t} = \frac{\partial \left(\frac{D \, \partial \theta}{\partial x} \right)}{\partial x} \tag{8.17}$$

Three general approaches are available for solving for D. The first assumes that D is a unique function of θ; in this case equation 8.17 remains in its current form. A second approach assumes steady flow (i.e., $(\partial \theta / \partial t) = 0$), in which case equation 8.17 is rewritten as

$$0 = \frac{\partial \left(\frac{D \, \partial \theta}{\partial x} \right)}{\partial x} \tag{8.18}$$

The third approach assumes that D is constant, and rewrites equation 8.17 as

$$\frac{\partial \theta}{\partial t} = D \left(\frac{\partial^2 \theta}{\partial x^2} \right) \tag{8.19}$$

Equation 8.19 is the best-known form of the diffusion equation, and is similar to the equation for heat flow or electron flow.

The most convenient way to measure D is in the laboratory, for which there are two commonly preferred methods. These include the pressure-plate outflow method (Gardner 1956), and the horizontal infiltration method (Bruce and Klute 1956), which uses the Boltzmann transformation and is discussed in the following section. In this section, we focus our discussion on the pressure-plate outflow method.

By placing a sample of a given soil in a pressure-plate apparatus, we subject it to a specific gas–phase pressure, for which the volume of water released from the sample may be recorded for each increment in pressure increase as a function of time. The pressure-plate apparatus typically is attached to a burette (used for outflow collection), and at time zero there is an initial gage pressure, P_i, for which the pressure in the apparatus and the water in the medium sample is in hydraulic equilibrium. Also at time zero, we apply a specific pressure ΔP, such that the final pressure in the apparatus, P_f, can be represented as $P_f = P_i + \Delta P$. The outflow volume collected in the burette is measured as a function of time. The smaller the pressure increments, ΔP, the more accurate the measurement of D and/or K. Remembering that $H = \psi_m + z$, then,

$$H = \frac{P}{\rho_f g} + z \tag{8.20}$$

where ρ_f is the fluid density (kg m^{-3}) and g is the gravitational constant. Since $P/\rho_f g \gg z$, z is safely neglected which results in the following non-linear differential equation:

$$\frac{\partial \theta}{\partial t} = \frac{1}{\rho_f g} \frac{\partial}{\partial z} \left[K(\theta) \frac{\partial P}{\partial z} \right] \tag{8.21}$$

Equation 8.21 is difficult to solve analytically, but if we linearize it, it is readily solved. We do this by assuming that the pressure increment, ΔP, is very small, yet large enough to obtain a

measurable volume of outflow. In so doing, K is constant during outflow and is moved to the outside of the differential operator. Additionally, by using this same process we assume that the water content and pressure have a linear relation during outflow, and that both fluid density and the gravitational factors are constant also. This allows us to write (skipping several steps in the math):

$$K(\theta)\left(-\frac{d\psi_m}{d\theta}\right) = \frac{K(\theta)}{\rho_f gb} = D(\theta) \tag{8.22}$$

where b represents the reciprocal of $-d\psi_m/d\theta$, assumed constant. Assuming D is constant during each pressure increment, we calculate D from the quantity of water released from a sample. Skipping about 25 mathematical steps, we can plot $\ln[Q_o - Q(t)]$ as a function of time. In this plot, Q_o represents the total amount of water released from the sample over a pressure increment ΔP, and $Q(t)$ is the quantity of outflow from the sample at time, t. Total outflow, Q_o, is expressed as $Q_o = bV\Delta P$, where V is the volume of the sample. Thus, in order to find D from the quantity of water collected from a sample, we use

$$\ln[Q_o - Q(t)] = \left[\ln\left(\frac{8Q_o}{\pi^2}\right)\right] - \left(\frac{\pi}{2L}\right)^2 Dt \tag{8.23}$$

where L is length of soil sample. The plot of $\ln[Q_o - Q(t)]$ versus time should yield a straight line with an intercept $\ln(8Q_o/\pi^2)$; S is slope of this line per unit time (see figure 8.7), given by

$$S = -\frac{\pi^2}{4L^2}D \tag{8.24}$$

which we rewrite to obtain

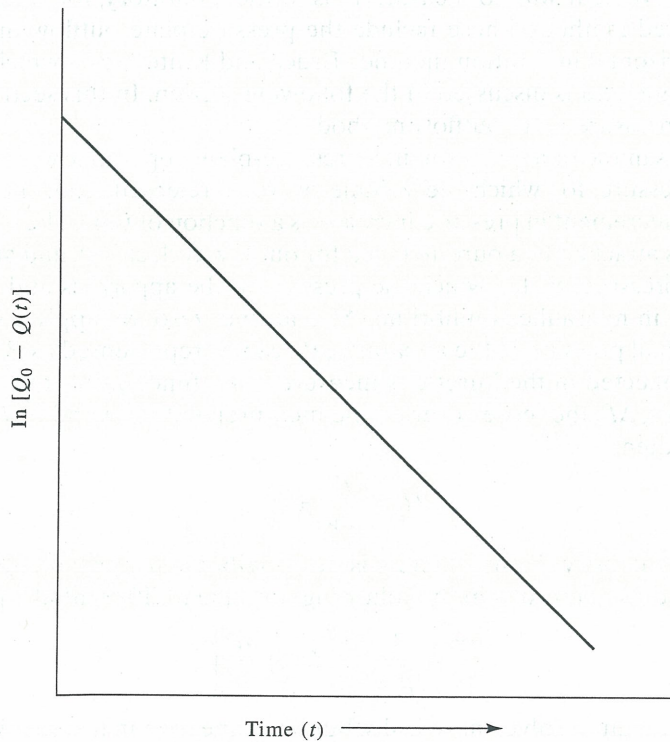$$D = S\left(\frac{4L^2}{\pi^2}\right) \tag{8.25}$$

Figure 8.7   Plot of $\ln[Q_o - Q(t)]$ versus time, t, to find D; where $\ln[Q_o - Q(t)] = [\ln(8Q_o/\pi^2)] - (\pi/2L)^2 Dt$ (after Gardner 1956)

For our theory, the plot is a straight line; however, experimental data often results in "tailing," as with a break-through curve (see chapter 10). Such behavior shows that the slope S and consequently, $D(\theta)$, is not constant over the range of pressure change ΔP; it does not mean that D or K is no longer a function of theta, since they depend on the total quantity of water outflow, Q_o, from the sample. Thus, the expression for capillary conductivity is written as

$$K_c = \frac{\rho_f \, g Q_o \, 4SL^2}{V \Delta P \pi^2} \tag{8.26}$$

where V is the volume of the sample. This is equation 16 of Gardner (1956), and allows us to solve for both K and D simultaneously from the same set of data, where the initially sought diffusivity D is given by equation 8.25. The constant, b, used in equation 8.24 is determined from the relation $b = (Q_o / V \Delta P)$.

It is of interest to note that this method was originally developed by Terzaghi (1943) to determine the diffusivity of soil samples during consolidation. Water is derived by compression of the sample due to the imposition of an incremental load, and the samples are saturated, but the theory and approach are the same.

8.5 BOLTZMANN'S TRANSFORMATION

The horizontal diffusivity equation is written as

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\left(\frac{\partial \theta}{\partial x}\right)\right] \tag{8.27}$$

However, equation 8.27 is a non-linear partial differential equation and cannot be solved by the usual methods. Assuming that θ is a single-valued function of a variable B (also a function of both time and distance), equation 8.27 is transformed to an ordinary differential equation through the use of Boltzmann's transformation (Boltzmann 1894; Crank 1956). The Boltzmann transformation is expressed as

$$B = \frac{x}{\sqrt{t}} \tag{8.28}$$

where x is distance (L) and t is time (T). The boundary conditions are

$$\theta = \theta_i, \quad \text{for} \quad B = \infty \quad (B \to \infty) \tag{8.29}$$

and

$$\theta = \theta_s, \quad \text{for} \quad B = 0 \tag{8.30}$$

These boundary conditions actually create a third condition because θ must be both continuous and differentiable (vary smoothly) with x, t, and also B, thus

$$\frac{d\theta}{dB} = 0, \quad \text{for} \quad \theta = \theta_i \tag{8.31}$$

These relations are seen in figure 8.8 using reduced measurements after Nielsen and Biggar (1962). By manipulating the above equations we obtain the differential equation of the curve in figure 8.8, and integrate it to find that

$$D(\theta_x) = \left(-\frac{1}{2\left(\dfrac{d\theta}{dB}\right)_{\theta_x}}\right)\int_{\theta_i}^{\theta_x} B \, d\theta \tag{8.32}$$

where θ_x is the volumetric water content at distance x, B can be obtained from equation 8.28, and θ_i is the initial volumetric-water content (preferably air-dry). One method to obtain $D(\theta_x)$ is to use a computer program. However, if you do not have such a program, the basics steps in obtaining $D(\theta_x)$ is to: (1) Plot θ versus x as in figure 8.8; (2) From this plot, evaluate $(d\theta/dB)_{\theta_x}$ and the integral of equation 8.32 at various values of θ_x. The value of the derivative $(d\theta/dB)$ is found by drawing tangents at various points along the curve ("eyeball it") and finding the slope, or semianalytically from the raw data. To evaluate the integral, simply divide the area under the curve of θ versus x into a finite number of different strips and find the approximate cumulative sum (area) of these strips; and (3) Determine D at the value of θ_x used in step 2 to obtain $D(\theta)$.

Since $D(\theta_x)$ only appears on the left side of equation 8.32, it is important to remember its relation to θ_x. Because θ_i is the lower limit in the integral, we have to integrate completely to the end of the tail in figure 8.8, that is, where the curve joins the x axis or the value of B as given in equation 8.28. As a result of this, the horizontal tube must be long enough so that the diffusing fluid front does not reach the end of the tube.

From the discussion above and the original experiment of Bruce and Klute (1956, 1962), we perceive that the Boltzmann transformation is valid for fluid movement in unsaturated soils for cases in which equation 8.27 is valid and where the following boundary conditions exist: $\theta(x,t) = \theta_i$ for $x > 0$, $t = 0$ and also $\theta(x,t) = \theta_s$ for $x = 0$, $t > 0$. Additionally, since $\theta = f(B)$, then equation 8.28 is expressed as
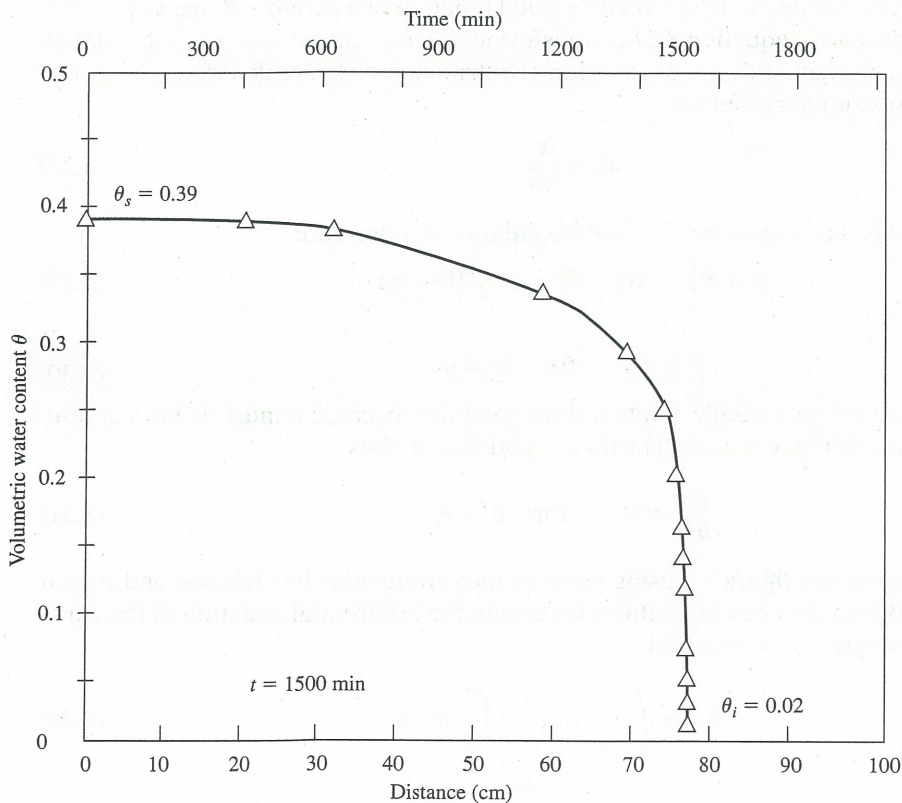
$$B(\theta) = \frac{x}{\sqrt{t}} \tag{8.33}$$

Figure 8.8 Plot of θ versus distance, x, to assist in obtaining diffusivity, $D(\theta_x)$, as described in equation 8.32 (data from Nielsen and Biggar 1962)

Simply, this implies that the quantity B as a function of θ is equal to the product on the right-hand side of equation 8.33. It should be noted that $B(\theta)$ versus θ will be different for different soils. Consequently, for positions along the horizontal column that have the same value of θ, then, despite distance x and time t, the product on the right-hand side of equation 8.33 is constant. If it is determined that θ at the wetting-front is constant, then we can plot distance of the front versus $\sqrt{t}$ at the front. The result should be a straight-line relation. However, if a straight line is not obtained, it usually indicates that the researcher failed to measure the portion of the wetting front that is at constant θ, not that the diffusion theory is invalid for the medium.

QUESTION 8.2

Volumetric-moisture content, θ, versus distance for a horizontal soil column of a sandy loam soil, for a time of 1500 minutes after water was applied at the input end ($x = 0$), is given in table 8.1. Utilize equation 8.33 to plot the Boltzmann transformation versus θ for this data.

Table 8.1

θ	Distance (cm)	θ	Distance (cm)
0.02	76.00	0.21	74.45
0.03	75.90	0.22	74.15
0.04	75.85	0.23	73.80
0.05	75.75	0.24	73.40
0.06	75.71	0.25	72.80
0.07	75.67	0.26	72.08
0.08	75.63	0.27	71.20
0.09	75.58	0.28	69.90
0.10	75.50	0.29	68.05
0.11	75.45	0.30	66.05
0.12	75.40	0.31	63.75
0.13	75.35	0.32	61.10
0.14	75.30	0.33	58.15
0.15	75.25	0.34	54.80
0.16	75.18	0.35	51.20
0.17	75.05	0.36	47.10
0.18	74.90	0.37	41.80
0.19	74.75	0.38	32.50
0.20	74.65	0.39	0

Source: Nielsen et al., 1962

SUMMARY

In this chapter we discussed the basic principles of water flow within an unsaturated medium; how, within the unsaturated zone the degree of saturation refers to the portion of pore volume filled with water, and that the unsaturated hydraulic conductivity decreases rapidly as volumetric water content decreases. Also, rather than a positive pressure head (as discussed in saturated fluid flow), flow in the unsaturated zone is primarily by a negative suction or matric potential. We also discussed the validity of Darcy's law for use in unsaturated flow. If a non-zero hydraulic gradient is applied to fluid in an unsaturated medium (no matter how small) and the water does not move or the velocity, v, is not a straight-line function of the hydraulic gradient, Darcy's law will not apply and the fluid is termed non-Newtonian in behavior. However, for most cases, Darcy's law is valid for describing water movement through unsaturated porous media. The extent to which non-Darcy flow exists in a soil is not fully understood.

We also discussed various physical and other flow factors such as stratified medium, the dependence of K on θ, and so on, which affect unsaturated hydraulic conductivity, and we developed the general flow equations for hydraulic conductivity and diffusivity. It was also discussed how to linearize the flow equation 8.27 for diffusivity by use of the Boltzmann transformation. This chapter is to serve as a beginning point for understanding the unsaturated flow concepts involving gaseous diffusion, contaminant migration, and moisture redistribution—all of which will be discussed in chapters 9, 10, and 11, respectively.

ANSWERS TO QUESTIONS

8.1. Using 8.12c, we obtain (remembering to convert from kPa to m for ψ): $K(\psi) = (16.72 \text{ m/day})/[1 + (2.044 \text{ m}/0.55 \text{ m})^4] = 0.087 \text{ m/day}$.

8.2. Using a spreadsheet program, we obtain Table 8.2, as well as the corresponding plot for the Boltzmann transformation.

Table 8.2

Moisture content (cm³/cm³)	Distance (cm)	Boltzmann[1] (cm/min^0.5)	Area at Point[2]	Cumulative area under curve[3]	Slope of curve at Point[4]	Diffusivity (cm²/min)[5]
0.02	76.00	1.9263	–	–	–	–
0.03	75.90	1.9597	0.0194	0.0194	0.2991	–0.0325
0.04	75.85	1.9584	0.0196	0.0390	–7.7460	0.0025
0.05	75.75	1.9559	0.0196	0.0586	–3.8730	0.0076
0.06	75.71	1.9548	0.0196	0.0781	–9.6825	0.0040
0.07	75.67	1.9538	0.0195	0.0977	–9.6825	0.0050
0.08	75.63	1.9528	0.0195	0.1172	–9.6825	0.0061
0.09	75.58	1.9515	0.0195	0.1367	–7.7460	0.0088
0.10	75.50	1.9494	0.0195	0.1562	–4.8412	0.0161
0.11	75.45	1.9481	0.0195	0.1757	–7.7460	0.0113
0.12	75.40	1.9468	0.0195	0.1952	–7.7460	0.0126
0.13	75.35	1.9455	0.0195	0.2147	–7.7460	0.0139
0.14	75.30	1.9442	0.0194	0.2341	–7.7460	0.0151
0.15	75.25	1.9429	0.0194	0.2536	–7.7460	0.0164
0.16	75.18	1.9411	0.0194	0.2730	–5.5328	0.0247
0.17	75.05	1.9378	0.0194	0.2924	–2.9792	0.0491
0.18	74.90	1.9339	0.0194	0.3117	–2.5820	0.0604
0.19	74.75	1.9300	0.0193	0.3310	–2.5820	0.0641
0.20	74.65	1.9275	0.0193	0.3503	–3.8730	0.0452
0.21	74.45	1.9223	0.0192	0.3696	–1.9365	0.0954
0.22	74.15	1.9145	0.0192	0.3888	–1.2910	0.1506
0.23	73.80	1.9055	0.0191	0.4079	–1.1066	0.1843
0.24	73.40	1.8952	0.0190	0.4269	–0.9682	0.2204
0.25	72.80	1.8797	0.0189	0.4457	–0.6455	0.3453
0.26	72.08	1.8611	0.0187	0.4645	–0.5379	0.4317
0.27	71.20	1.8384	0.0185	0.4829	–0.4401	0.5487
0.28	69.90	1.8048	0.0182	0.5012	–0.2979	0.8411
0.29	68.05	1.7570	0.0178	0.5190	–0.2094	1.2395
0.30	66.05	1.7054	0.0173	0.5363	–0.1936	1.3847
0.31	63.75	1.6460	0.0168	0.5530	–0.1684	1.6421
0.32	61.10	1.5776	0.0161	0.5692	–0.1462	1.9472
0.33	58.15	1.5014	0.0154	0.5846	–0.1313	2.2262
0.34	54.80	1.4149	0.0146	0.5991	–0.1156	2.5912
0.35	51.20	1.3220	0.0137	0.6128	–0.1076	2.8481
0.36	47.10	1.2161	0.0127	0.6255	–0.0945	3.3109

(*contd.*)

Table 8.2 (*continued*)

Moisture content (cm³/cm³)	Distance (cm)	Boltzmann[1] cm/min^0.5	Area[2] at Point	Cumulative[3] area under curve	Slope[4] of curve at Point	Diffusivity[5] cm²/min
0.37	41.80	1.0793	0.0115	0.6370	−0.0731	4.3585
0.38	32.50	0.8391	0.0096	0.6466	−0.0416	7.7630
0.39	0.00	0.0000	0.0042	0.6508	−0.0119	27.3049

[1]Distance ÷ $\sqrt{t}$, where $t = 1500$ min
[2]0.01 ((1.9263 + 1.9597)/2)
[3]Cumulative sum
[4]Current θ − Prev. θ/Boltzmann, that is, $(0.03 - 0.02)/(1.9597 - 1.9263)$
[5](1/slope)*(−0.5)*Cumulative Area; here = 0.0194
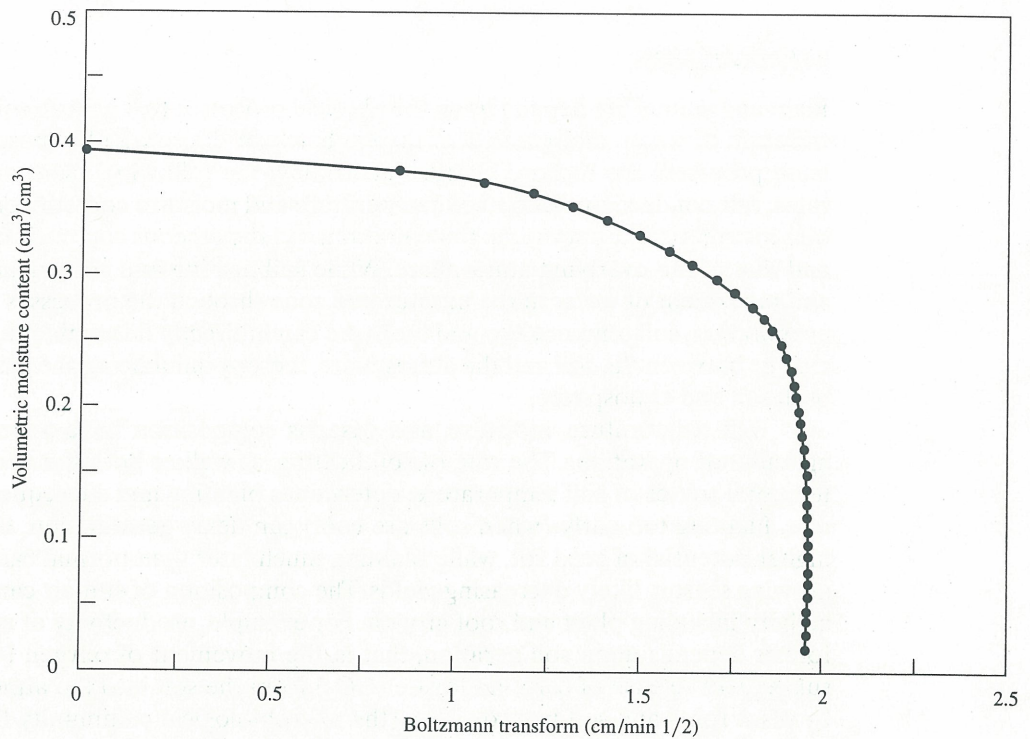
ADDITIONAL QUESTIONS

8.3. You are given two soils, a loam and a sand. At the same θ, the loam has a lower hydraulic conductivity than the sand. Explain this.

8.4. Why can a soil have different K values for a specific h_m?

8.5. Suppose $\partial H/\partial z$ is negative, positive, and zero. What is the direction of flow for each case?

8.6. You have a soil with the condition $h < 0$ at the surface. Would you expect a water layer to be present on the surface?

8.7. We can express $D = K / (d\theta/dP_m) = K (dP_m/d\theta)$, where P_m is the pressure equivalent of the matric potential. With this in mind, is D a function of the volume fraction of water?